Ch5 Lecture 1

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

An **eigenvector** of A is a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) such that for some scalar λ, we have

$$A\mathbf{x} = \lambda\mathbf{x}$$

. . .

Why do we care? One reason: matrix multiplication can become *scalar* multiplication:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^2\mathbf{x} &= A(A\mathbf{x}) = A\lambda\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x} \\ &\vdots \\ A^k\mathbf{x} &= A\left(A^{k-1}\mathbf{x}\right) = \cdots = \lambda^k\mathbf{x}. \end{aligned}$$

. . .

Will see how to use these ideas to understand dynamics even for vectors that aren't eigenvectors.

Finding Eigenvalues and Eigenvectors

λ is an eigenvalue of A if $A\mathbf{x} = \lambda\mathbf{x}\,(= \lambda I\mathbf{x})$

. . .

$$\mathbf{0} = \lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x}$$

. . .

pause

. . .

So, λ is an eigenvalue of $A \iff \det(\lambda I - A) = 0$.

. . .

If A is a square $n \times n$ matrix, the equation $\det(\lambda I - A) = 0$ is called the **characteristic equation** of A, and the n th-degree monic polynomial $p(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial** of A.

Eigenspaces and Eigensystems

The **eigenspace** corresponding to eigenvalue λ is the subspace $\mathcal{N}(\lambda I - A)$ of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

We write $\mathcal{E}_\lambda(A) = \mathcal{N}(\lambda I - A)$.

. . .

Note: $\mathbf{0}$ is in every eigenspace, but it is not an eigenvector.

. . .

The **eigensystem** of the matrix A is a list of all the eigenvalues of A and, for each eigenvalue λ, a complete description of its eigenspace – usually a basis.

. . .

Eigensystem Algorithm:

Given $n \times n$ matrix A, to find an eigensystem for A :

(1) Find the eigenvalues of A.

(2) For each scalar λ in (1), use the null space algorithm to find a basis of the eigenspace $\mathcal{N}(\lambda I - A)$.

A reminder: the Null Space algorithm

Given an $m \times n$ matrix A.

1. Compute the reduced row echelon form R of A.
2. Use R to find the general solution to the homogeneous system $A\mathbf{x} = 0$.

. . .

3. Write the general solution $\mathbf{x} = (x_1, x_2, \dots, x_n)$ to the homogeneous system in the form

. . .

$$\mathbf{x} = x_{i_1}\mathbf{w}_1 + x_{i_2}\mathbf{w}_2 + \cdots + x_{i_{n-r}}\mathbf{w}_{n-r}$$

where $x_{i_1}, x_{i_2}, \dots, x_{i_{n-r}}$ are the free variables.

. . .

4. List the vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n-r}$. These form a basis of $\mathcal{N}(A)$.

Example of the Null Space algorithm

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 3 & 2 & 2 & 5 \\ 3 & 4 & 1 & 4 \end{bmatrix}$$

. . .

The reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which becomes, in vector notation,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}.$$

:::

Example

Find an eigensystem for the matrix $A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$

. . .

$$\begin{aligned} 0 = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 7 & -4 \\ -3 & \lambda - 6 \end{bmatrix} \\ = (\lambda - 7)(\lambda - 6) - (-3)(-4) \\ = \lambda^2 - 13\lambda + 42 - 12 \\ = \lambda^2 - 13\lambda + 30 \\ = (\lambda - 3)(\lambda - 10). \end{aligned}$$

. . .

Eigenvalues are $\lambda = 3$ and $\lambda = 10$.

. . .

For $\lambda = 3$, we have

$$A - 3I = \begin{bmatrix} 7-3 & 4 \\ 3 & 3-3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \xrightarrow[E_{21}(-3/4)]{E_1(1/4)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

. . .

. . .

Thus, a basis of $\mathcal{E}_3(A)$ is $\{(-1, 1)\}$.

. . .

Similarly, for $\lambda = 10$, the basis of $\mathcal{E}_{10}(A)$ is $\{(4/3, 1)\}$

Two examples with repeated eigenvalues

What happens when an eigenvalue is repeated?

Example: find the eigenspace for eigenvalue $\lambda = 2$ of these two matrices:

$$(a) \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad (b) \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

. . .

Both matrices have eigenvalues $\lambda = 1, 2, 2$, but their eigenspaces for $\lambda = 2$ are different:

. . .

After row reducing the first matrix, we get

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{21}(1)]{\longrightarrow} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

. . .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

. . .

A basis for $\mathcal{E}_2(A)$ is $\{(1,0,0), (0,-2,1)\}$.

Dimension of $\mathcal{E}_2(A) = 2$.

$$(a) \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad (b) \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

For the second matrix, we have

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{21}(1)]{\longrightarrow} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_2(1/2)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

. . .

Only free variable is x_1, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

. . .

A basis for $\mathcal{E}_2(A)$ is $\{(1,0,0)\}$. Dimension of $\mathcal{E}_2(A) = 1$.

Eigenvalue Multiplicity

The **algebraic multiplicity** of λ is the multiplicity of λ as a root of the characteristic equation. The **geometric multiplicity** of λ is the dimension of the space $\mathcal{E}_\lambda(A) = \mathcal{N}(\lambda I - A)$.

. . .

The eigenvalue λ of A is said to be **simple** if its algebraic multiplicity is 1 , that is, the number of times it occurs as a root of the characteristic equation is 1 . Otherwise, the eigenvalue is said to be **repeated**.

. . .

For an $n \times n$ matrix A, the sum of the algebraic multiplicities of all the eigenvalues of A is n.

(In other words, there are n eigenvalues, counting algebraic multiplicities and complex numbers).

The *geometric multiplicity* of an eigenvalue is always less than or equal to its *algebraic multiplicity.*

Defective Matrices

A matrix is **defective** if one of its eigenvalues has geometric multiplicity less than its algebraic multiplicity.

. . .

This means that the sum of the geometric multiplicities of all the eigenvalues is less than n.

Similarity and Diagonalization

Functions of Diagonal Matrices

It's easy to find functions of matrices if the matrix is **diagonal**.

$$D = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

. . .

$D^k = \operatorname{diag}\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\}$

. . .

$$f(D) = \operatorname{diag}\{f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\}$$

Similar Matrices

A matrix A is said to be **similar** to matrix B if there exists an invertible matrix P such that

$$P^{-1}AP = B.$$

P is called a **similarity transformation matrix**.

. . .

If A is similar to M, say $P^{-1}AP = M$. Then:

(1) For every polynomial $q(x)$,
$$q(M) = P^{-1}q(A)P.$$

(2) The matrices A and M have the same eigenvalues.

. . .

So in particular,

$$A^k = PD^kP^{-1}$$

Diagonalization

. . .

To diagonalize an $n \times n$ matrix A, we just need to find n independent eigenvectors and put them into the columns of a matrix P. Then $P^{-1}AP = D$

Example

Find $f(A)$ for $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ with $f(x) = \sin\left(\frac{\pi}{2}x\right)$

. . .

From before, the basis vectors for the eigenspace of $\lambda = 2$ are $\{(1, 0, 0), (0, -2, 1)\}$

The basis for the eigenspace of $\lambda = 1$ is $\{(-1, 1, 0)\}$

. . .

$$P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

. . .

$$P^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

. . .

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$$\sin\left(\frac{\pi}{2}A\right) = P^{-1}\sin\left(\frac{\pi}{2}D\right)P$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sin\left(2\frac{\pi}{2}\right) & 0 & 0 \\ 0 & \sin\left(2\frac{\pi}{2}\right) & 0 \\ 0 & 0 & \sin\left(1\frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

When are matrices diagonalizable?

The $n \times n$ matrix A is diagonalizable if and only if there exists a linearly independent set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ of A ... in which case $P = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$ is a diagonalizing matrix for A.

. . .

If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Caution: Just because the $n \times n$ matrix A has fewer than n distinct eigenvalues, you may not conclude that it is not diagonalizable.

Application to Discrete Dynamical Systems

Diagonalizable Transition Matrices

Suppose you have the discrete dynamical system

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$$

. . .

Assume A is diagonalizable, then we can write complete, linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Suppose the eigenvalues are ordered, so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

. . .

Spectral Radius and Dominant Eigenvalue

$$\mathbf{x}^{(k)} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

. . .

The **spectral radius** $\rho(A)$ of a matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is

$$\rho(A) = \max\left\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\right\}.$$

. . .

If $|\lambda_k| = \rho(A)$ and λ_k is the **only** eigenvalue with this property, then λ_k is the **dominant eigenvalue** of A.

Implications of the Spectral Radius

For $n \times n$ transition matrix A, initial state vector $\mathbf{x}^{(0)}$:

(1) If $\rho(A) < 1$, then $\lim_{k\to\infty} A^k\mathbf{x}^{(0)} = \mathbf{0}$.

(2) If $\rho(A) = 1$, then the sequence of norms $\left\{\left\|\mathbf{x}^{(k)}\right\|\right\}_{k=0}^{\infty}$ is bounded.

(3) If $\rho(A) = 1$ and $\lambda = 1$ is the dominant eigenvalue of A, then $\lim_{k\to\infty}\mathbf{x}^{(k)}$ is an element of $\mathcal{E}_1(A)$ – either an eigenvector or $\mathbf{0}$.

(4) If $\rho(A) > 1$, then for some choices of $\mathbf{x}^{(0)}$ we have $\lim_{k\to\infty}\|\mathbf{x}\| = \infty$

Example

You have too many frogs.

. . .

There is a species of bird that eats frogs. (Yay!) Can these help you with your frog problem?

. . .

Let F_k and B_k be the number of frogs and birds at year k.

$$\begin{aligned} B_{k+1} &= 0.6B_k + 0.4F_k, \\ F_{k+1} &= -rB_k + 1.4F_k, \end{aligned}$$

where r is the rate at which the birds eat the frogs. (Suppose it's around 0.35).

. . .

Is it a good idea to introduce these birds?

Let the population vector in the k th year be $\mathbf{x}^{(k)} = (B_k, F_k)$

$$\begin{bmatrix} B_{k+1} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ -0.35 & 1.4 \end{bmatrix} \begin{bmatrix} B_k \\ F_k \end{bmatrix}$$

. . .

Find eigenvalues and eigenvectors of the transition matrix.

```python
import numpy as np
A = np.array([[0.6, 0.4], [-0.35, 1.4]])
eigvals, eigvecs = np.linalg.eig(A)
eigvals_sorted = np.sort(eigvals)[::-1]
eigvecs_sorted = eigvecs[:, np.argsort(eigvals)[::-1]]
print('Eigenvalues: ' + str(eigvals_sorted))
print('Eigenvectors: '+ str(eigvecs_sorted))
```

```
Eigenvalues: [1.14142136 0.85857864]
Eigenvectors: [[-0.59421716 -0.83980458]
 [-0.80430465 -0.54288882]]
```

Check: use Sympy to diagonalize the matrix.

```python
import sympy as sp
#| echo: true
Asym = sp.Matrix(A)
P,D = Asym.diagonalize()


print("D : {}".format(D))
print("P : {}".format(P))

P*D*P.inv()
```

```
D : Matrix([[0.858578643762691, 0], [0, 1.14142135623731]])
P : Matrix([[0.839804577036026, 0.635068381284203], [0.542888821389189, 0.859598960745815]])
```

$$\begin{bmatrix} 0.6 & 0.4 \\ -0.35 & 1.4 \end{bmatrix}$$

```python
x0 = np.array([100, 10000])

def print_iterations(x0, A, n1=10, n2=1):
  x = x0
  for i in range(n1):
    for j in range(n2):
      x = A @ x
    print(x)


print_iterations(x0, A)
```

```
[ 4060. 13965.]
[ 8022. 18130.]
[12065.2 22574.3]
[16268.84 27381.2 ]
[20713.784 32639.586]
[25484.1048 38445.596 ]
[30668.70128 44904.39772]
[36362.979856 52132.11136 ]
[42670.6324576 60257.9129544]
[49705.54465632 69426.356776  ]
```

Different starting points...

```python
x0 = eigvecs_sorted[0]*-1000
print_iterations(x0, A)
```

```
[692.45212576 967.75040247]
[ 802.57143644 1112.49231944]
[ 926.53978964 1276.58924446]
[1066.55957157 1462.93601587]
```

```
[1225.11014929 1674.81457217]
[1404.99191844 1915.95184879]
[1609.37589058 2190.58541685]
[1841.85970109 2503.53802188]
[2106.5310294  2860.30233525]
[2408.03955174 3267.13740907]
```

What if we start with the second eigenvector?

```
x0 = eigvecs_sorted[1]*-1000
print_iterations(x0, A)
```

```
[699.73831812 478.5377227 ]
[611.25807995 425.04440043]
[536.77260814 381.12183262]
[474.51229793 345.70015282]
[422.98743989 317.90090967]
[380.9528278  297.01566957]
[347.37796451 282.48844767]
[321.42215777 273.90153916]
[302.41391033 270.96439961]
[289.83410604 273.50529084]
```

. . .

Looks good so far, doesn't blow up. But if we continue (only printing results every 10 years now):

```
x0 = eigvecs_sorted[1]*-1000
print_iterations(x0, A,n1 = 10, n2 = 10)
```

```
[289.83410604 273.50529084]
[493.86429346 642.61221892]
[1724.50444871 2328.57998379]
[6445.12974686 8722.60198143]
[24186.97763221 32738.09888286]
[ 90789.4036837  122888.24713326]
[340796.1924577  461285.82916599]
[1279247.72362158 1731530.09096614]
```

```
[4801916.36748821 6499650.17996028]
[18024969.23527191 24397758.22361141]
```

. . .

Even the roundoff errors are starting to make the frog population explode

What about complex eigenvalues?

Let's repeat the above for a different matrix:

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

. . .

```python
A = np.array([[0, 2], [-2, 0]])
eigvals, eigvecs = np.linalg.eig(A)
eigvals_sorted = np.sort(eigvals)[::-1]
eigvecs_sorted = eigvecs[:, np.argsort(eigvals)[::-1]]
print('Eigenvalues: ' + str(eigvals_sorted))
print('Eigenvectors: '+ str(eigvecs_sorted))
print_iterations(np.array([1,2]), A,n1 = 10, n2 = 1)
```

```
Eigenvalues: [0.+2.j 0.-2.j]
Eigenvectors: [[0.        -0.70710678j 0.        +0.70710678j]
 [0.70710678+0.j         0.70710678-0.j        ]]
[ 4 -2]
[-4 -8]
[-16   8]
[16 32]
[ 64 -32]
[ -64 -128]
[-256  128]
[256 512]
[1024 -512]
[-1024 -2048]
```

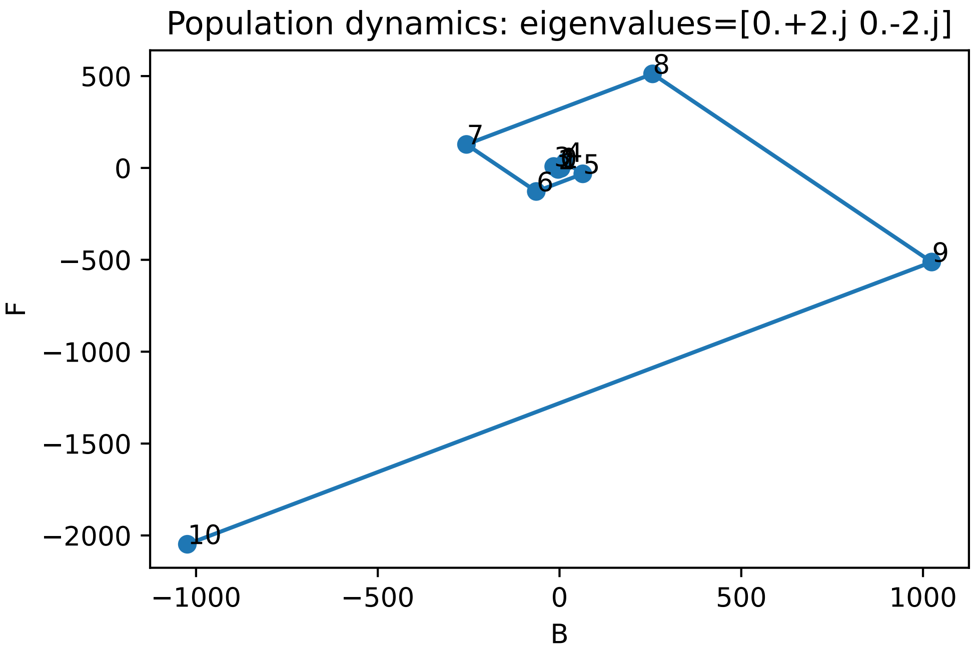
. . .

Let's plot that…

```
import matplotlib.pyplot as plt

def plot_iterations(x0, A, n1=10, n2=1):
  plt.clf()
  x = x0
  x_vals = [x]
  for i in range(n1):
    for j in range(n2):
      x = A @ x
    x_vals.append(x)
  x_vals = np.array(x_vals)
  plt.plot(x_vals[:,0], x_vals[:,1], 'o-',)
  for i, txt in enumerate(range(len(x_vals))):
    plt.annotate(txt, (x_vals[i,0]+1, x_vals[i,1]+1))
  #plt.plot(x_vals[:,0], x_vals[:,1], 'o-')
  plt.xlabel('B')
  plt.ylabel('F')
  plt.title('Population dynamics: eigenvalues='+str(np.linalg.eig(A)[0]))
  plt.show()
```

```
plot_iterations(np.array([1,2]), np.array([[0, 2], [-2, 0]]))
```



Your turn

```
import matplotlib.pyplot as plt
import numpy as np
def plot_iterations(x0, A, n1=10, n2=1):
  plt.clf()
  x = x0
  x_vals = [x]
  for i in range(n1):
    for j in range(n2):
      x = A @ x
    x_vals.append(x)
  x_vals = np.array(x_vals)
  plt.plot(x_vals[:,0], x_vals[:,1], 'o-',)
  for i, txt in enumerate(range(len(x_vals))):
    plt.annotate(txt, (x_vals[i,0]+1, x_vals[i,1]+1))
  #plt.plot(x_vals[:,0], x_vals[:,1], 'o-')
  plt.xlabel('B')
  plt.ylabel('F')
  plt.title('Population dynamics: eigenvalues='+str(np.linalg.eig(A)[0]))
  plt.show()
```

```
A = np.array([[0, 2], [-2, 0]])
plot_iterations(np.array([1,2]), A)
```

Another example

Figure 1: Fibonacci Statue

. . .

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, …

$F_0 = 0$ and $F_1 = 1$. For $n \geq 2$, the n-th Fibonacci number is

$$F_n = F_{n-1} + F_{n-2}$$

. . .

What is the n-th Fibonacci number? What is F_{100}?

(Treatment from here)

Get this into matrix form.

Let

$$\bar{x}_k = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$$

. . .

Then

$$\bar{x}_{k+1} = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \bar{x}_k$$

for $\bar{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

. . .

To understand the dynamics, we need to diagonalize the matrix.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \\ \lambda &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Eigenvalues (call them phi and psi):

$$\begin{aligned} \varphi &= \frac{1+\sqrt{5}}{2} \approx 1.618 \\ \psi &= \frac{1-\sqrt{5}}{2} \approx -0.618 \end{aligned}$$

Eigenvectors:

$$\bar{v}_\varphi = \begin{bmatrix} 1 \\ \varphi \end{bmatrix} \quad \text{and} \quad \bar{v}_\psi = \begin{bmatrix} 1 \\ \psi \end{bmatrix}$$

. . .

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ \varphi & \psi \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix}$$

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ \varphi & \psi \end{bmatrix} \begin{bmatrix} \varphi^k & 0 \\ 0 & \psi^k \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix} \\ &= \begin{bmatrix} \varphi^k & \psi^k \\ \varphi^{k+1} & \psi^{k+1} \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi\varphi^k + \varphi\psi^k & \varphi^k - \psi^k \\ -\psi\varphi^{k+1} + \varphi\psi^{k+1} & \varphi^{k+1} - \psi^{k+1} \end{bmatrix} \end{aligned}$$

Finally,

$$\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = A^k \bar{x}_0 = \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi\varphi^k + \varphi\psi^k & \varphi^k - \psi^k \\ -\psi\varphi^{k+1} + \varphi\psi^{k+1} & \varphi^{k+1} - \psi^{k+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^k - \psi^k \\ \varphi^{k+1} - \psi^{k+1} \end{bmatrix}$$

. . .

and so

$$F_k = \frac{1}{\sqrt{5}} \left(\varphi^k - \psi^k\right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k + \left(\frac{1-\sqrt{5}}{2} \right)^k \right)$$

. . .

But for large k, ψ^k is very small, so

$$F_k \approx \frac{\varphi^k}{\sqrt{5}}$$

. . .

$$\lim_{k \to \infty} \frac{F_{k+1}}{F_k} = \varphi$$

This is the golden ratio!

Go back to some of the previous examples we've used, or that you've used in your projects. Work in pairs to find the eigenvalues and eigenvectors of the transition matrix. What do the eigenvalues tell you about the dynamics of the system?