# Review

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## Chapter 1: Linear Systems

### Linear Systems (ch1Lecture1b, ch1Lecture2)

- Applications of linear systems
- Putting linear systems in matrix form
- *Gauss-Jordan to get to row echelon form
- *Solving linear systems with augmented matrices
- Free vs bound variables
- Ill-conditioned systems & rounding errors

Getting to row echelon form:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -3 & -9 \end{bmatrix} \overrightarrow{E_2(-1/3)} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix} \overrightarrow{E_{12}(-1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

### Augmented matrix to solve linear system:

$$\begin{aligned} z &= 2 \\ x + y + z &= 2 \\ 2x + 2y + 4z &= 8 \end{aligned}$$

. . .

Augmented matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 4 & 8 \end{bmatrix} \overset{E_{12}}{\longrightarrow} \begin{bmatrix} (1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 2 & 2 & 4 & 8 \end{bmatrix} \xrightarrow[E_{31}(-2)]{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We keep on going...

$$\begin{bmatrix} (1) & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[E_2(1/2)]{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\overline{E_{32}(-1)} \begin{bmatrix} (1) & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[E_{12}(-1)]{\longrightarrow} \begin{bmatrix} (1) & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There's still no information on $y$.

$$x = -y$$
$$z = 2$$
$$y \text{ is free.}$$

# *Chapter 2: Matrices*

## *Matrix multiplication (ch2Lecture1)*

- Matrix-vector multiplication as a linear combination of columns
- Matrix multiplication as an operation
- *Scaling, rotation, shear

### *Scaling and rotation*

To rotate a vector by $\theta$:

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

. . .

Scaling:

$$A = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}$$

Shearing: adding a constant shear factor times one coordinate to another coordinate of the point.

$$A = \begin{bmatrix} 1 & s_2 \\ s_1 & 1 \end{bmatrix}$$

## Graphs and Directed Graphs (ch2Lecture2)

- *Adjacency and incidence matrices
- Degree of a vertex
- *Paths and cycles
- PageRank

### Adjacency and incidence matrices

**Adjacency matrix**: A square matrix whose $(i, j)$ th entry is the number of edges going from vertex $i$ to vertex $j$

**Incidence matrix**: A matrix whose rows correspond to vertices and columns correspond to edges. The $(i, j)$ th entry is 1 if vertex $i$ is the tail of edge $j$, -1 if it is the head, and 0 otherwise.

### Paths and cycles

- Number of paths of length $n$ from vertex $i$ to vertex $j$ is the $(i, j)$ th entry of the matrix $A^n$.
- Vertex power is the sum of the entries in the $ith$ row of $A + A^2$.
- In an the incidence matrix of a digraph which is a cycle, every row must sum to zero.

## Discrete Dynamical Systems (Ch2Lecture2)

- Transition matrices
- *Markov chains

### *Markov Chains*

A **distribution vector** is a vector whose entries are nonnegative and sum to 1.

A **stochastic matrix** is a square matrix whose columns are distribution vectors.

A **Markov chain** is a discrete dynamical system whose initial state $\mathbf{x}^{(0)}$ is a distribution vector and whose transition matrix $A$ is stochastic, i.e., each column of $A$ is a distribution vector.

## *Difference Equations (Ch2Lecture3)*

- *Difference equations in matrix form
- *Examples

### *Difference Equation in Matrix Form*

From HW2: put ths difference equation in matrix form:

$$y_{k+2} - y_{k+1} - y_k = 0$$

. . .

Steps: 1. Make two equations. Solve for $y_{k+2}$: $y_{k+2} = y_{k+1} + y_k$. Also of course $y_{k+1} = y_{k+1}$. 2. Define the vector $\mathbf{y}^{(k)} = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$ 3. Put the two equations in matrix form: . . .

$\begin{bmatrix} y_{k+1} \\ y_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$

### *Examples of difference equations*

Reaction-diffusion:

$$a_{x,t+1} = a_{x,t} + dt \left( \frac{D_a}{dx^2} (a_{x+1,t} + a_{x-1,t} - 2a_{x,t}) \right)$$

Heat in a rod:

$$-y_{i-1} + 2y_i - y_{i+1} = \frac{h^2}{K} f\left(x_i\right)$$

## MCMC (Ch2 lecture 4)

- MCMC
  - Simulate a distribution using a Markov chain
  - Sample from the simulated distribution
- Restricted Boltzmann Machines

## Inverses and determinants (Ch2 lecture 4 & 5)

- Inverse of a matrix
- *Determinants
- Singularity
- LU factorization

### Determinants

The **determinant** of a square $n \times n$ matrix $A = \left[a_{ij}\right]$, $\det A$, is defined recursively:

If $n = 1$ then $\det A = a_{11}$;

otherwise,

- suppose we have determinents for all square matrices of size less than $n$
- Define $M_{ij}(A)$ as the determinant of the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $i$ th row and $j$ th column of $A$

then

$$\begin{aligned}
\det A &= \sum_{k=1}^{n} a_{k1}(-1)^{k+1} M_{k1}(A) \\
&= a_{11}M_{11}(A) - a_{21}M_{21}(A) + \cdots + (-1)^{n+1} a_{n1} M_{n1}(A)
\end{aligned}$$

### Laws of Determinants

- D1: If $A$ is an upper triangular matrix, then the determinant of $A$ is the product of all the diagonal elements of $A$.
- D2: If $B$ is obtained from $A$ by multiplying one row of $A$ by the scalar $c$, then $\det B = c \cdot \det A$.
- D3: If $B$ is obtained from $A$ by interchanging two rows of $A$, then $\det B = -\det A$.

- D4: If $B$ is obtained from $A$ by adding a multiple of one row of $A$ to another row of $A$, then $\det B = \det A$.
- D5: The matrix $A$ is invertible if and only if $\det A \neq 0$.
- D6: Given matrices $A, B$ of the same size, $\det AB = \det A \det B$.
- D7: For all square matrices $A$, $\det A^T = \det A$

## Chapter 3: Vector Spaces

### Spaces of matrices (ch3 lecture 1)

- Basis
- Fundamental subspaces:
  - Column space
  - Null space
  - Row space
- Rank
- Consistency

#### Column and Row Spaces

The **column space** of the $m \times n$ matrix $A$ is the subspace $\mathcal{C}(A)$ of $\mathbb{R}^m$ spanned by the columns of $A$.

The **row space** of the $m \times n$ matrix $A$ is the subspace $\mathcal{R}(A)$ of $\mathbb{R}^n$ spanned by the transposes of the rows of $A$

. . .

A basis for the column space of $A$ is the set of pivot columns of $A$. (Find these by row reducing $A$ and choosing the columns with leading 1s)

#### Null Space

The **null space** of the $m \times n$ matrix $A$ is the subset $\mathcal{N}(A)$ of $\mathbb{R}^n$

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

. . .

$\mathcal{N}(A)$ is just the solution set to $A\mathbf{x} = \mathbf{0}$

. . .

For example, if $A$ is invertible, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

. . .

so $\mathcal{N}(A)$ is just $\{\mathbf{0}\}$.

. . .

$A$ is invertible exactly if $\mathcal{N}(A) = \{\mathbf{0}\}$

### *Finding a basis for the null space*

Given an $m \times n$ matrix $A$.

1. Compute the reduced row echelon form $R$ of $A$.
2. Use $R$ to find the general solution to the homogeneous system $A\mathbf{x} = 0$.

. . .

3. Write the general solution $\mathbf{x} = (x_1, x_2, \dots, x_n)$ to the homogeneous system in the form

. . .

$$\mathbf{x} = x_{i_1}\mathbf{w}_1 + x_{i_2}\mathbf{w}_2 + \cdots + x_{i_{n-r}}\mathbf{w}_{n-r}$$

where $x_{i_1}, x_{i_2}, \dots, x_{i_{n-r}}$ are the free variables.

. . .

4. List the vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n-r}$. These form a basis of $\mathcal{N}(A)$.

### *Using the Null Space*

- The general solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_h$ is in the null space of $A$.
- The null space of $A$ is orthogonal to the row space of $A$ (or the column space of $A^T$. The dot product of any vector in the null space of $A$ with any vector in the row space of $A$ is 0.)

# Chapter 4: Geometrical Aspects of Standard Spaces

## Orthogonality (ch4 lecture 1 and 2)

- Geometrical intuitions
- *Least Squares & Normal equations
- Finding orthogonal bases (Gram-Schmidt)

### Least Squares and Normal Equations

To find the least squares solution to $A\mathbf{x} = \mathbf{b}$, we minimize the squared error $\|A\mathbf{x} - \mathbf{b}\|^2$ by solving the Normal Equations for $\mathbf{x}$:

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

### QR Factorization

If $A$ is an $m \times n$ full-column-rank matrix, then $A = QR$, where the columns of the $m \times n$ matrix $Q$ are orthonormal vectors and the $n \times n$ matrix $R$ is upper triangular with nonzero diagonal entries.

1. Start with the columns of A, $A = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$. (For now assume they are linearly independent.)
2. Do Gram-Schmidt on the columns of $A$ to get orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$.

. . .

$$A = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \begin{bmatrix} 1 & \frac{\mathbf{q}_1 \cdot \mathbf{w}_2}{\mathbf{q}_1 \cdot \mathbf{q}_1} & \frac{\mathbf{q}_1 \cdot \mathbf{w}_3}{\mathbf{q}_1 \cdot \mathbf{q}_1} \\ 0 & 1 & \frac{\mathbf{q}_2 \cdot \mathbf{w}_3}{\mathbf{q}_2 \cdot \mathbf{q}_2} \\ 0 & 0 & 1 \end{bmatrix}$$

# Chapter 5: Eigenvalues and Eigenvectors

## Eigenvalues and Eigenvectors (ch5 lecture 1)

- Definition
- *How to find them
- Similarity and Diagonalization
- Applications to dynamical systems
- Spectral radius

#### Finding Eigenvalues and Eigenvectors

If $A$ is a square $n \times n$ matrix, the equation $\det(\lambda I - A) = 0$ is called the **characteristic equation** of $A$

The eigenvalues of $A$ are the roots of the characteristic equation.

For each scalar $\lambda$ in (1), use the null space algorithm to find a basis of the eigenspace $\mathcal{N}(\lambda I - A)$.

### Symmetric matrices (ch5 lecture 2)

- Properties of symmetric matrices
- Quadratic forms

### SVD (ch5 lecture 3, 4)

- Definition
- *Psuedoinverse
- Applications to least squares
- Image compression

#### Pseudoinverse

The **pseudoinverse** of $A$ is $A^+ = VS^+U^T$

$S^+$ is the matrix with the reciprocals of the non-zero singular values on the diagonal, and zeros elsewhere.

. . .

Can find least squares solutions:

$$
\begin{aligned}
Ax &= b \\
x &\approx A^+b
\end{aligned}
$$

### PCA (ch5 lecture 5)

- Definition
- Applications to data analysis

## *Chapter 6: Fourier Transform*

### *Fourier Transform (ChNone, Ch6 lecture 1)*