

# Ch2 Lecture 5

## ***Today: Rewriting Matrix Expressions***

1. **Blocking:** Group entries to simplify multiplication
2. **Transpose:** Switch row/column viewpoints
3. **Determinants:** Extract a single number that answers “invertible?”

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### **i** Running Example

We'll use the system  $2x_1 - x_2 = 1$ ,  $4x_1 + 4x_2 = 20$  throughout to see how each tool helps us understand and solve it.

Matrix:  $A = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$

## ***Block multiplication***

### ***Block Multiplication***

**Difficulty: Large multiplications are painful**

Suppose we need to multiply these matrices...

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

pause

## Concept: What is Block Multiplication?

The key idea: **Partition** so the block column-widths of the left matrix match the block row-heights of the right matrix.

Then you multiply *as if each block were an entry* (but using matrix multiplication inside – you'll see what that means in a moment).

## Method: Block Multiplication

### Steps to perform block multiplication

1. **Partition** so the block column-widths of the left matrix match the block row-heights of the right matrix.
2. **Multiply block-rows by block-columns** (same pattern as ordinary matrix multiplication).

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**Formula:** If

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad N = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

then

$$MN = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}.$$

### Worked example: block-multiply the matrices from before

We started with the (painful) product

$$\left[ \begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array} \right] \left[ \begin{array}{cc|cc} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

The vertical/horizontal lines are the “rectangles” that tell us where the **blocks** are.

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We can give these blocks labels:

$$M = \left[ \begin{array}{c|c} A & 0_{2 \times 2} \\ \hline 0_{1 \times 2} & B \end{array} \right], \quad N = \left[ \begin{array}{c|c} 0_{2 \times 2} & C \\ \hline 0_{2 \times 2} & I_2 \end{array} \right].$$

**Name the blocks**

$$M = \left[ \begin{array}{c|c} A & 0_{2 \times 2} \\ \hline 0_{1 \times 2} & B \end{array} \right], \quad N = \left[ \begin{array}{c|c} 0_{2 \times 2} & C \\ \hline 0_{2 \times 2} & I_2 \end{array} \right].$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Multiply block-by-block (like “row times column”)**

$$M = \left[ \begin{array}{c|c} A & 0_{2 \times 2} \\ \hline 0_{1 \times 2} & B \end{array} \right], \quad N = \left[ \begin{array}{c|c} 0_{2 \times 2} & C \\ \hline 0_{2 \times 2} & I_2 \end{array} \right].$$

Each block of  $(MN)$  is a block-row times a block-column:

$$MN = \left[ \begin{array}{c|c} A \cdot 0_{2 \times 2} + 0_{2 \times 2} \cdot 0_{2 \times 2} & A \cdot C + 0_{2 \times 2} \cdot I_2 \\ \hline 0_{1 \times 2} \cdot 0_{2 \times 2} + B \cdot 0_{2 \times 2} & 0_{1 \times 2} \cdot C + B \cdot I_2 \end{array} \right].$$

...

Now simplify using  $(0 \cdot 0) = 0$  and  $(B \cdot I_2 = B)$ :

$$MN = \left[ \begin{array}{c|c} 0_{2 \times 2} & AC \\ \hline 0_{1 \times 2} & B \end{array} \right].$$

$$MN = \left[ \begin{array}{c|c} 0_{2 \times 2} & AC \\ \hline 0_{1 \times 2} & B \end{array} \right].$$

Compute the only nontrivial block:

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 7 \end{bmatrix}.$$

So the full product is

$$MN = \left[ \begin{array}{cc|cc} 0 & 0 & 4 & 3 \\ 0 & 0 & 10 & 7 \\ \hline 0 & 0 & 1 & 0 \end{array} \right].$$

**Key insight:** In our example, lots of blocks are **zeros** and one block is an **identity**, so most terms vanish or simplify immediately.

### **Column Vectors as Blocks**

The most important example of blocking: view a matrix as blocked into its columns.

$$Ax = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3$$

This shows  $Ax$  as a **linear combination of columns** — an important interpretation that connects matrix multiplication to the column space (we'll see this in Chapter 3, day 8).

Blocking isn't just a trick:

- **Expose real structure:** When matrices have zeros/identities/block-diagonal (or triangular) form, there are parts that **don't interact**, so big products/solves break into smaller ones (many terms become (0)).
- **Algorithm notation:** many methods are “work on a submatrix, update the rest.” Blocks keep dimensions straight; this shows up in **QR** (and later eigenvalue iterations) as “top-left / bottom-right” reasoning.

### **Skills: what you should be able to do**

- Identify compatible block partitions
- Multiply block-rows by block-columns
- Recognize the ‘columns-as-blocks’ identity:  $Ax = \sum x_i \mathbf{a}_i$

## Transpose and Conjugate Transpose

### Concept: Why Transpose Matters

The transpose operation unlocks fundamental tools:

- **Inner products:**  $\mathbf{u}^T \mathbf{v}$  is the dot product—we'll see this is fundamental in least squares (Chapter 4, day 9-10)
- **Symmetric matrices:** The transpose operation lets us define symmetry in a matrix. Symmetry appears everywhere—quadratic forms (which we'll see today), and many applications

## Transpose and Conjugate Transpose

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### Definition: Transpose and Conjugate Transpose

- Let  $A = [a_{ij}]$  be an  $m \times n$  matrix with (possibly) complex entries.
- The **transpose** of  $A$  is the  $n \times m$  matrix  $A^T$  obtained by interchanging the rows and columns of  $A$
- The **conjugate** of  $A$  is the matrix  $\bar{A} = [\bar{a}_{ij}]$
- Finally, the **conjugate (Hermitian) transpose** of  $A$  is the matrix  $A^* = \bar{A}^T$ .

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#### 💡 Running Example

For our system with  $A = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}$ :

$$A^T = \begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix}$$

Notice how rows become columns and vice versa.

### Example 1

Find the transpose and conjugate transpose of:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

...

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$$

...

Because the matrix is real, it has no complex entries, so the conjugate transpose is the same as the transpose.

### Example 2

Find the transpose and conjugate transpose of:

$$\begin{bmatrix} 1 & 1+i \\ 0 & 2i \end{bmatrix}$$

...

$$\begin{bmatrix} 1 & 1+i \\ 0 & 2i \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 1+i & 2i \end{bmatrix}, \quad \begin{bmatrix} 1 & 1+i \\ 0 & 2i \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 1-i & -2i \end{bmatrix}$$

### Method: Transpose Laws

Let  $A$  and  $B$  be matrices of the appropriate sizes so that the following operations make sense, and  $c$  a scalar.

The following laws hold:

- $(A + B)^T = A^T + B^T$  (distribute over addition)
- $(AB)^T = B^T A^T$  (reverse order!)
- $(cA)^T = cA^T$  (scalar comes out)
- $(A^T)^T = A$  (involution)

...

**Key warning:** Product order reverses:  $(AB)^T = B^T A^T$ , not  $A^T B^T$ !

## **Symmetric and Hermitian Matrices**

The matrix  $A$  is said to be:

- **symmetric** if  $A^T = A$
- **Hermitian** if  $A^* = A$

### **Examples of Symmetric, Hermitian, and Neither**

Let's see what these look like in concrete  $2 \times 2$  examples.

#### **1. Symmetric (real entries, $A^T = A$ ):**

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

Here,  $A^T = A$ . Since the entries are real,  $A$  is both symmetric and Hermitian.

#### **2. Hermitian (complex entries, $A^* = A$ , but not symmetric):**

$$B = \begin{bmatrix} 1 & 2+i \\ 2-i & 4 \end{bmatrix}$$

Here,  $B$  is not symmetric because  $(2+i) \neq (2-i)$ , but it is Hermitian because

$$B^* = \overline{B}^T = \begin{bmatrix} 1 & 2-i \\ 2+i & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2+i \\ 2-i & 4 \end{bmatrix} = B.$$

#### **3. Neither symmetric nor Hermitian:**

$$C = \begin{bmatrix} 0 & 1+i \\ 3 & 2 \end{bmatrix}$$

Here,  $C^T = \begin{bmatrix} 0 & 3 \\ 1+i & 2 \end{bmatrix} \neq C$  and  $C^* = \begin{bmatrix} 0 & 3 \\ 1-i & 2 \end{bmatrix} \neq C$ , so  $C$  is neither symmetric nor Hermitian.

### **Check: Symmetric vs Hermitian**

Is this matrix symmetric? Hermitian?

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \quad \dots$$

It's Hermitian, but not symmetric.

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}^* = \begin{bmatrix} 1 & \overline{1+i} \\ \overline{1-i} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

and then

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$$

### **Skills: what you should be able to do**

- Compute transpose and conjugate transpose
- Apply  $(AB)^T = B^T A^T$  fluently (watch the order!)
- Recognize symmetric vs Hermitian matrices

## **Inner and outer products**

### **Concept: Inner product**

- Let  $\mathbf{u}$  and  $\mathbf{v}$  be column vectors of the same size, say  $n \times 1$ .
- Then the **inner product** of  $\mathbf{u}$  and  $\mathbf{v}$  is the scalar quantity  $\mathbf{u}^T \mathbf{v}$

Find the inner product of

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\mathbf{u}^T \mathbf{v} = [2, -1, 1] \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 2 \cdot 3 + (-1)4 + 1 \cdot 1 = 3$$

### Concept: Outer product

- The **outer product** of  $\mathbf{u}$  and  $\mathbf{v}$  is the  $n \times n$  matrix  $\mathbf{u}\mathbf{v}^T$ .

Find the outer product of

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} [3, 4, 1] = \begin{bmatrix} 2 \cdot 3 & 2 \cdot 4 & 2 \cdot 1 \\ -1 \cdot 3 & -1 \cdot 4 & -1 \cdot 1 \\ 1 \cdot 3 & 1 \cdot 4 & 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 2 \\ -3 & -4 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

### Applications of Inner and Outer Products

**Inner product** ( $\mathbf{u}^T\mathbf{v}$ ): - Dot product: measures similarity/alignment between vectors - Projections: fundamental in least squares (we'll see in Chapter 4, day 9-10) - PCA: we'll see Principal Component Analysis in Chapter 5 (day 14)

**Outer product** ( $\mathbf{u}\mathbf{v}^T$ ): - Rank-1 matrices: building blocks for matrix factorizations - Low-rank approximations: SVD and data compression (we'll see in Chapter 5, day 14)

### Example: Inner products in our running example

#### 💡 Running Example

The columns of our matrix  $A$  are:

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Their inner product:  $\mathbf{a}_1^T \mathbf{a}_2 = 2(-1) + 4(4) = 14$

## From Products to Quadratic Forms

Inner and outer products show how matrix multiplication can produce either a scalar (inner) or a matrix (outer).

**Quadratic forms** are a major application: they package many terms into a single scalar expression using transposes.

## Concept: Quadratic Forms

A **quadratic form** is a homogeneous polynomial of degree 2 in  $n$  variables. For example,

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$$Q(x, y, z) = x^2 + 2y^2 + z^2 + 2xy + yz + 3xz.$$

...

We can express this in matrix form!

$$\begin{aligned} x(x + 2y + 3z) + y(2y + z) + z^2 &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x + 2y + 3z \\ 2y + z \\ z \end{bmatrix} \\ &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x}^T A \mathbf{x} \end{aligned}$$

## Example: Rewrite as $\mathbf{x}^T A \mathbf{x}$

### Physics example: kinetic energy

For a particle moving in the plane, the velocity vector is  $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ . With mass  $m$ , the kinetic energy is the quadratic polynomial

$$T = \frac{1}{2}m(v_x^2 + v_y^2).$$

...

Write this as a quadratic form using the identity matrix:

$$T = \frac{1}{2} (mv_x^2 + mv_y^2) = \frac{1}{2} \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \frac{1}{2} \mathbf{v}^T (mI_2) \mathbf{v}.$$

### **What does the quadratic form buy us?**

- **One formula, any dimension:**  $T = \frac{1}{2} \mathbf{v}^T (mI) \mathbf{v}$  works in 2D, 3D, or for longer vectors.
- **The matrix encodes the physics:**  $mI$  means “same weight for movement in every direction.” More generally,  $T = \frac{1}{2} \mathbf{v}^T M \mathbf{v}$  lets  $M$  encode different weights/couplings.
- **Immediate properties:** if  $M$  is symmetric positive definite, then  $T \geq 0$  for all  $\mathbf{v}$ .

### **Skills: what you should be able to do**

- Compute  $\mathbf{u}^T \mathbf{v}$  and interpret as dot product/similarity
- Compute  $\mathbf{u}\mathbf{v}^T$  and recognize it as a rank-1 matrix
- Rewrite a quadratic polynomial as  $\mathbf{x}^T A \mathbf{x}$

## **Determinants**

### **Difficulty: Invertible or not?**

We've seen that solving  $A\mathbf{x} = \mathbf{b}$  requires checking if  $A$  is invertible.

**Question:** Is there a single number that tells us whether  $A$  is invertible?

**Answer:** Yes! The **determinant** of  $A$ .

### **Concept: Why Determinants?**

#### **Invertibility**

- If  $\det A \neq 0$ , then  $A$  is invertible
- If  $\det A = 0$ , then  $A$  is singular (not invertible)

...

## Why determinants matter beyond invertibility

- **Eigenvalues:** Product of eigenvalues equals determinant (we'll see eigenvalues in Chapter 5, day 11)
- **Explicit formulas:** For  $2 \times 2$  matrices, we have simple explicit formulas (inverse and Cramer's rule) that depend on the determinant

...  
**Computational note:** Gaussian elimination is usually better for computation. For larger matrices, explicit formulas exist but are computationally inefficient. The  $2 \times 2$  case is the exception where explicit formulas are practical.

### 💡 Running Example

For our system with  $A = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}$ :

We'll compute  $\det A = 12 \neq 0$ , confirming that  $A$  is invertible and our system has a unique solution.

## Definition of the determinant

The **determinant** of a square  $n \times n$  matrix  $A = [a_{ij}]$ ,  $\det A$ , is defined recursively:

If  $n = 1$  then  $\det A = a_{11}$ ;

...

otherwise,

- suppose we have determinants for all square matrices of size less than  $n$
- Define  $M_{ij}(A)$  as the determinant of the  $(n - 1) \times (n - 1)$  matrix obtained from  $A$  by deleting the  $i$  th row and  $j$  th column of  $A$

...

then

$$\begin{aligned}\det A &= \sum_{k=1}^n a_{k1}(-1)^{k+1} M_{k1}(A) \\ &= a_{11}M_{11}(A) - a_{21}M_{21}(A) + \cdots + (-1)^{n+1}a_{n1}M_{n1}(A)\end{aligned}$$

## **Laws of the determinant**

Determinant of an upper-triangular matrix:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = \cdots = a_{11} \cdot a_{22} \cdots a_{nn}.$$

...

D1: If  $A$  is an upper triangular matrix, then the determinant of  $A$  is the product of all the diagonal elements of  $A$ .

## **More Laws of Determinants**

- D2: If  $B$  is obtained from  $A$  by multiplying one row of  $A$  by the scalar  $c$ , then  $\det B = c \cdot \det A$ .
- D3: If  $B$  is obtained from  $A$  by interchanging two rows of  $A$ , then  $\det B = -\det A$ .
- D4: If  $B$  is obtained from  $A$  by adding a multiple of one row of  $A$  to another row of  $A$ , then  $\det B = \det A$ .

## **Determinant Laws in terms of Elementary Matrices**

- D2:  $\det(E_i(c)A) = c \cdot \det A$  (remember that for  $E_i(c)$  to be an elementary matrix,  $c \neq 0$ ).
- D3:  $\det(E_{ij}A) = -\det A$ .
- D4:  $\det(E_{ij}(s)A) = \det A$ .

## **Determinant of Row Echelon Form**

Let  $R$  be the reduced row echelon form of  $A$ , obtained through multiplication by elementary matrices:

$$R = E_1 E_2 \cdots E_k A.$$

...

Determinant of both sides:

$$\det R = \det(E_1 E_2 \cdots E_k A) = \pm(\text{nonzero constant}) \cdot \det A.$$

Therefore,  $\det A = 0$  **precisely** when  $\det R = 0$ .

- $R$  is upper triangular, so  $\det R$  is the product of the diagonal entries of  $R$ .
- If  $\text{rank } A < n$ , then there will be **zeros** in some of the diagonal entries, so  $\det R = 0$ .
- If  $\text{rank } A = n$ , the diagonal entries are all 1, so  $\det R = 1$ .
  - A square matrix with rank  $n$  is **invertible**

Therefore,

D5: The matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

### ***Two more Determinant Laws***

D6: Given matrices  $A, B$  of the same size,

$$\det AB = \det A \det B.$$

(but beware,  $\det A + \det B \neq \det(A + B)$ )

D7: For all square matrices  $A$ ,  $\det A^T = \det A$

### ***Method: Compute determinant in practice***

#### ***Steps to compute the determinant***

1. Use elementary row operations to get the matrix into upper triangular form
2. Keep track of row operations to adjust for sign changes and scalar multiplications
3. Multiply the diagonal entries

### Key rules

- Row swap: multiply by  $-1$
- Row scale by  $c$ : multiply by  $c$
- Row replacement: no change

## Explicit Formulas for inverses and solving linear systems with $2 \times 2$ Matrices

Method ( $2 \times 2$  toolkit):

- To find inverses, use the  $2 \times 2$  inverse formula:  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- To solve linear systems, apply Cramer's rule for  $2 \times 2$  systems

**$2 \times 2$  Inverse Formula:**

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 4 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

This is a useful explicit formula for  $2 \times 2$  matrices! :::

...

### Why Only $2 \times 2$ ?

For larger matrices, explicit formulas exist (using cofactors and adjoints), but they are computationally inefficient compared to Gaussian elimination. The  $2 \times 2$  case is the exception where the explicit formula is simple and practical.

### Cramer's Rule ( $2 \times 2$ )

For  $n \times n$  systems, Cramer's rule gives us an explicit formula for the solution. But here we will only consider the  $2 \times 2$  case.

Let  $A$  be an invertible  $2 \times 2$  matrix and  $\mathbf{b}$  a  $2 \times 1$  column vector.

Denote by  $B_1$  the matrix obtained from  $A$  by replacing the first column of  $A$  by  $\mathbf{b}$ , and  $B_2$  the matrix obtained by replacing the second column.

Then the linear system  $A\mathbf{x} = \mathbf{b}$  has unique solution:

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}$$

### **Example: Using Cramer's Rule**

Solve the system:

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ 4x_1 + 4x_2 &= 20 \end{aligned}$$

The coefficient matrix and right-hand side are:

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$$

Compute  $\det A = 2 \cdot 4 - (-1) \cdot 4 = 8 + 4 = 12$ .

Now apply Cramer's rule:

$$x_1 = \frac{\det B_1}{\det A} = \frac{\begin{vmatrix} 1 & -1 \\ 20 & 4 \end{vmatrix}}{12} = \frac{4 - (-20)}{12} = \frac{24}{12} = 2$$

$$x_2 = \frac{\det B_2}{\det A} = \frac{\begin{vmatrix} 2 & 1 \\ 4 & 20 \end{vmatrix}}{12} = \frac{40 - 4}{12} = \frac{36}{12} = 3$$

Check:  $2(2) - 3 = 1$  and  $4(2) + 4(3) = 20$

### **Summary of Laws of Determinants**

Let  $A, B$  be  $n \times n$  matrices.

- D1: If  $A$  is upper triangular,  $\det A$  is the product of all the diagonal elements of  $A$ .
- D2:  $\det(E_i(c)A) = c \cdot \det A$ .
- D3:  $\det(E_{ij}A) = -\det A$ .
- D4:  $\det(E_{ij}(s)A) = \det A$ .
- D5: The matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

- D6:  $\det AB = \det A \det B$ .
- D7:  $\det A^T = \det A$ .

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***Skills: what you should be able to do***

- Decide invertibility using  $\det A$  (invertible iff  $\det A \neq 0$ )
- Compute  $\det A$  efficiently via elimination (tracking row swaps/scales)
- Use the  $2 \times 2$  inverse formula when needed
- Solve a  $2 \times 2$  system using Cramer's rule