Appendix D

Glossary: A Dictionary for Linear Algebra

Adjacency matrix of a graph Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ for an undirected graph.

Affine transformation $T(v) = Av + v_0 =$ linear transformation plus shift.

Associative Law $(AB)C = A(BC)$ Parentheses can be removed to leave ABC.

Augmented matrix $[A\ \ b]$ $Ax = b$ is solvable when b is in the column space of A; then $[A\ \ b]$ has the same rank as A. Elimination on $[A\ \ b]$ keeps equations correct.

Back substitution Upper triangular systems are solved in reverse order x_n to x_1.

Basis for V Independent vectors $v_1, \dots, v_d$ whose linear combinations give every v in **V**. A vector space has many bases!

Big formula for n **by** n **determinants** $\det(A)$ is a sum of $n!$ terms, one term for each permutation P of the columns. That term is the product $a_{1\alpha} \cdots a_{n\omega}$ down the diagonal of the reordered matrix, times $\det(P) = \pm 1$.

Block matrix A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns.

Block multiplication of AB is allowed if the block shapes permit (the columns of A and rows of B must be in matching blocks).

Cayley-Hamilton Theorem $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix*.

Change of basis matrix M The old basis vectors v_j are combinations $\sum m_{ij} w_i$ of the new basis vectors. The coordinates of $c_1 v_1 + \cdots + c_n v_n = d_1 w_1 + \cdots + d_n w_n$ are related by $d = Mc$. (For $n = 2$, set $v_1 = m_{11} w_1 + m_{21} w_2$, $v_2 = m_{12} w_1 + m_{22} w_2$.)

Characteristic equation $\det(A - \lambda I) = 0$ The n roots are the eigenvalues of A.

Cholesky factorization $A = CC^{\mathrm{T}} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0I + c_1S + \cdots + c_{n-1}S^{n-1}$. $Cx =$ **convolution** $c * x$. Eigenvectors in F.

Cofactor C_{ij} Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $Ax = b$ The vector b becomes a combination of the columns of A. The system is solvable only when b is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ Space of all combinations of the columns of A.

Commuting matrices $AB = BA$ If diagonalizable, they share n eigenvectors.

Companion matrix Put $c_1, \ldots, c_n$ in row n and put $n-1$ 1s along diagonal 1. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots)$.

Complete solution $x = x_p + x_n$ **to** $Ax = b$ (Particular x_p) + (x_n in nullspace).

Complex conjugate $\bar{z} = a - ib$ for any complex number $z = a + ib$.Then $z\bar{z} = |z|^2$.

Condition number $\operatorname{cond}(A) = \kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$ In $Ax = b$, the relative change $\|\delta x\|/\|x\|$ is less than $\operatorname{cond}(A)$ times the relative change $\|\delta b\|/\|b\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method A sequence of steps to solve positive definite $Ax = b$ by minimizing $\frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$ over growing Krylov subspaces.

Covariance matrix Σ When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of x_ix_j. With means $\bar{x}_i$, the matrix Σ = mean of $(x - \bar{x})(x - \bar{x})^{\mathrm{T}}$ is positive (semi)definite; it is diagonal if the x_i are independent.

Cramer's Rule for $Ax = b$ B_j has b replacing column j of A, and $x_j = |B_j|/|A|$.

Cross product $u \times v$ **in** $\mathbf{R}^3$ Vector perpendicular to u and v, length $\|u\|\|v\||\sin\theta| =$ parallelogram area, computed as the "determinant" of $[i\ \ j\ \ k;\ \ u_1\ \ u_2\ \ u_3;\ \ v_1\ \ v_2\ \ v_3]$.

Cyclic shift S Permutation with $s_{21} = 1$, $s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$ Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$, $|A^{-1}| = 1/|A|$, and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n-1$, volume of box $= |\det(A)|$.

Diagonal matrix D $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda$ = eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$ $\Lambda =$ eigenvalue matrix and $S =$ eigenvector matrix. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(\mathbf{V}) =$ number of vectors in any basis for $\mathbf{V}$.

Distributive Law $A(B+C) = AB+AC$ Add then multiply, or multiply then add.

Dot product $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$ Complex dot product is $\bar{x}^{\mathrm{T}}y$. Perpendicular vectors have zero dot product. $(AB)_{ij} =$ (row i of A)$\cdot$(column j of B).

Echelon matrix U The first nonzero entry (the pivot) in each row comes after the pivot in the previous row. All zero rows come last.

Eigenvalue λ and eigenvector x $Ax = \lambda x$ with $x \neq 0$, so $\det(A - \lambda I) = 0$.

Eigshow Graphical 2 by 2 eigenvalues and singular values (MATLAB or Java).

Elimination A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathrm{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix $=$ **Elementary matrix** E_{ij} The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $x^{\mathrm{T}}Ax = 1$ A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|x\| = 1$ the vectors $y = Ax$ lie on the ellipse $\|A^{-1}y\|^2 = y^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}u(0)$ solves $u' = Au$.

Factorization $A = LU$ If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT) A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n x$ and $F_n^{-1}c$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers 0, 1, 1, 2, 3, 5,... satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1+\sqrt{5})/2$ the largest eigenvalue of the Fibonacci matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Four fundamental subspaces of A $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{T}})$, $\boldsymbol{N}(A^{\mathrm{T}})$.

Fourier matrix F Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}}F = nI$. Then $y = Fc$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A Columns without pivots; combinations of earlier columns.

Free variable x_i Column i has no pivot in elimination. We can give the $n-r$ free variables any values, then $Ax=b$ determines the r pivot variables (if solvable!).

Full column rank $r=n$ Independent columns, $\boldsymbol{N}(A)=\{0\}$, no free variables.

Full row rank $r=m$ Independent rows, at least one solution to $Ax=b$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements (perpendicular subspaces of $\mathbf{R}^n$ with dimensions r and $n-r$) from $Ax=0$. Applied to A^{T}, the column space $\boldsymbol{C}(A)$ is the orthogonal complement of $\boldsymbol{N}(A^{\mathrm{T}})$.

Gauss-Jordan method Invert A by row operations on $[A\ \ I]$ to reach $[I\ \ A^{-1}]$.

Gram-Schmidt orthogonalization $A=QR$ Independent columns in A, orthonormal columns in Q. Each column q_j of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\operatorname{diag}(R)>0$.

Graph G Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n-1$ edges and no closed loops. A **directed graph** has a direction arrow specified on each edge.

Hankel matrix H Constant along each antidiagonal; h_{ij} depends on $i+j$.

Hermitian matrix $A^{\mathrm{H}}=\overline{A}^{\mathrm{T}}=A$ Complex analog of a symmetric matrix: $\overline{a_{ji}}=a_{ij}$.

Hessenberg matrix H Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix $\operatorname{hilb}(n)$ Entries $H_{ij}=1/(i+j-1)=\int_0^1 x^{i-1}x^{j-1}dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number.

Hypercube matrix P_L^2 Row $n+1$ counts corners, edges, faces, ..., of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**)** Diagonal entries $=1$, off-diagonal entries $=0$.

Incidence matrix of a directed graph The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $v_1,\ldots,v_k$ No combination $c_1v_1+\cdots+c_kv_k=$ zero vector unless all $c_i=0$. If the v's are the columns of A, the only solution to $Ax=0$ is $x=0$.

Inverse matrix A^{-1} Square matrix with $A^{-1}A=I$ and $AA^{-1}=I$. No inverse if $\det A=0$ and $\operatorname{rank}(A)<n$, and $Ax=0$ for a nonzero vector x. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$ Cofactor formula $(A^{-1})_{ij}=C_{ji}/\det A$.

Iterative method A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$ If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \text{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1s on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector $(1, 0, \ldots, 0)$.

Kirchhoff's Laws *Current law*: net current (in minus out) is zero at each node. *Voltage law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$ Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, b)$ The subspace spanned by $b, Ab, \ldots, A^{j-1}b$. Numerical methods approximate $A^{-1}b$ by x_j with residual $b - Ax_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least-squares solution $\widehat{x}$ The vector $\widehat{x}$ that minimizes the error $\|e\|^2$ solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Then $e = b - A\widehat{x}$ is orthogonal to all columns of A.

Left inverse A^+ If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ Nullspace of A^{T} = "left nullspace" of A because $y^{\mathrm{T}}A = 0^{\mathrm{T}}$.

Length $\|x\|$ Square root of $x^{\mathrm{T}}x$ (Pythagoras in n dimensions).

Linear combination $cv + dw$ **or** $\sum c_j v_j$ Vector addition and scalar multiplication.

Linear transformation T Each vector v in the input space transforms to $T(v)$ in the output space, and linearity requires $T(cv + dw) = cT(v) + dT(w)$. Examples: Matrix multiplication Av, differentiation in function space.

Linearly dependent $v_1, \ldots, v_n$ A combination other than all $c_i = 0$ gives $\sum c_i v_i = 0$.

Lucas numbers $L = 2, 1, 3, 4, \ldots$, satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_n^2$, with eigenvalues $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ of the Fibonacci matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Compare $L_0 = 2$ with Fibonacci.

Markov matrix M All $m_{ij} \geq 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady-state eigenvector $Ms = s > 0$.

Matrix multiplication AB The i, j entry of AB is (row i of A) $\cdot$ (column j of B) $= \sum a_{ik}b_{kj}$. By columns: column j of $AB = A$ times column j of B. By rows: row i of A multiplies B. Columns times rows: AB = sum of (column k)(row k). All these equivalent definitions come from the rule that AB times x equals A times Bx.

Minimal polynomial of A The lowest-degree polynomial with $m(A)$ = zero matrix. The roots of m are eigenvalues, and $m(\lambda)$ divides $\det(A - \lambda I)$.

Multiplication $Ax = x_1(\text{column } 1) + \cdots + x_n(\text{column } n)$ = combination of columns.

Multiplicities AM **and** GM The algebraic multiplicity AM of an eigenvalue λ is the number of times λ appears as a root of $\det(A-\lambda I)=0$. The geometric multiplicity GM is the number of independent eigenvectors (= dimension of the eigenspace for λ).

Multiplier ℓ_{ij} The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: $\ell_{ij}=$ (entry to eliminate)/(jth pivot).

Network A directed graph that has constants $c_1,\ldots,c_m$ associated with the edges.

Nilpotent matrix N Some power of N is the zero matrix, $N^k=0$. The only eigenvalue is $\lambda=0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$ **of a matrix** The "ℓ^2 norm" is the maximum ratio $\|Ax\|/\|x\|=\sigma_{\max}$. Then $\|Ax\|\le\|A\|\|x\|$, $\|AB\|\le\|A\|\|B\|$, and $\|A+B\|\le\|A\|+\|B\|$. **Frobenius norm** $\|A\|_F^2=\sum\sum a_{ij}^2$; ℓ^1 and ℓ^∞ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^{\mathrm{T}}A\widehat{x}=A^{\mathrm{T}}b$ Gives the least-squares solution to $Ax=b$ if A has full rank n. The equation says that (columns of A) $\cdot\,(b-A\widehat{x})=0$.

Normal matrix N $NN^{\mathrm{T}}=N^{\mathrm{T}}N$, leads to orthonormal (complex) eigenvectors.

Nullspace matrix N The columns of N are the $n-r$ special solutions to $As=0$.

Nullspace $\boldsymbol{N}(A)$ Solutions to $Ax=0$. Dimension $n-r=$ (# columns) $-$ rank.

Orthogonal matrix Q Square matrix with orthonormal columns, so $Q^{\mathrm{T}}Q=I$ implies $Q^{\mathrm{T}}=Q^{-1}$. Preserves length and angles, $\|Qx\|=\|x\|$ and $(Qx)^{\mathrm{T}}(Qy)=x^{\mathrm{T}}y$. All $|\lambda|=1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces Every v in $\mathbf{V}$ is orthogonal to every w in $\mathbf{W}$.

Orthonormal vectors $q_1,\ldots,q_n$ Dot products are $q_i^{\mathrm{T}}q_j=0$, if $i\neq j$ and $q_i^{\mathrm{T}}q_j=1$. The matrix Q with these orthonormal columns has $Q^{\mathrm{T}}Q=I$. If $m=n$, then $Q^{\mathrm{T}}=Q^{-1}$ and $q_1,\ldots,q_n$ is an **orthonormal basis** for $\mathbf{R}^n$: every $v=\sum(v^{\mathrm{T}}q_j)q_j$.

Outer product is uv^{T} column times row = rank-1 matrix.

Partial pivoting In elimination, the jth pivot is chosen as the largest available entry (in absolute value) in column j. Then all multipliers have $|\ell_{ij}|\le 1$. Roundoff error is controlled (depending on the *condition number* of A).

Particular solution x_p Any solution to $Ax=b$; often x_p has free variables $=0$.

Pascal matrix $P_S=\mathrm{pascal}(n)$ The symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S=P_LP_U$ all contain Pascal's triangle with det $=1$ (see index for more properties).

Permutation matrix P There are $n!$ orders of $1,\ldots,n$; the $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is a product of row exchanges P_{ij}; P is *even* or *odd* ($\det P = 1$ or -1) based on the number of exchanges.

Pivot columns of A Columns that contain pivots after row reduction; not combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot d The first nonzero entry when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$ Solutions to $a^{\mathrm{T}}x = 0$ give the plane (dimension $n-1$) perpendicular to $a \neq 0$.

Polar decomposition $A = QH$ Orthogonal Q, positive (semi)definite H.

Positive definite matrix A Symmetric matrix with positive eigenvalues and positive pivots. Definition: $x^{\mathrm{T}}Ax > 0$ unless $x = 0$.

Projection matrix P **onto subspace** S Projection $p = Pb$ is the closest point to b in $\mathbf{S}$, error $e = b - Pb$ is perpendicular to $\mathbf{S}$. $P^2 = P = P^{\mathrm{T}}$, eigenvalues are 1 or 0, eigenvectors are in $\mathbf{S}$ or $\mathbf{S}^{\perp}$. If columns of A = basis for $\mathbf{S}$, then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Projection $p = a(a^{\mathrm{T}}b/a^{\mathrm{T}}a)$ **onto the line through** a $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ has rank 1.

Pseudoinverse A^+ **(Moore-Penrose inverse)** The n by m matrix that "inverts" A from column space back to row space, with $\boldsymbol{N}(A^+) = \boldsymbol{N}(A^{\mathrm{T}})$. A^+A and AA^+ are the projection matrices onto the row space and column space. $\mathrm{rank}(A^+) = \mathrm{rank}(A)$.

Random matrix $\mathrm{rand}(n)$ **or** $\mathrm{randn}(n)$ MATLAB creates a matrix with random entries, uniformly distributed on $[0\ \ 1]$ for rand, and standard normal distribution for randn.

Rank 1 matrix $A = uv^{\mathrm{T}} \neq 0$ Column and row spaces = lines cu and cv.

Rank $r(A)$ Equals number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(x) = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ For $A = A^{\mathrm{T}}$, $\lambda_{\min} \leq q(x) \leq \lambda_{\max}$. Those extremes are reached at the eigenvectors x for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form $R = \mathrm{rref}(A)$ Pivots= 1; zeros above and below pivots; r nonzero rows of R give a basis for the row space of A.

Reflection matrix $Q = I - 2uu^{\mathrm{T}}$ The unit vector u is reflected to $Qu = -u$. All vectors x in the plane $u^{\mathrm{T}}x = 0$ are unchanged because $Qx = x$. The "Householder matrix" has $Q^{\mathrm{T}} = Q^{-1} = Q$.

Right inverse A^+ If A has full row rank m, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the plane by θ, and $R^{-1} = R^{\mathrm{T}}$ rotates back by $-\theta$. Orthogonal matrix, eigenvalues $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors $(1, \pm i)$.

Row picture of $Ax = b$ Each equation gives a plane in $\mathbf{R}^n$ planes intersect at x.

Row space $\boldsymbol{C}(A^{\mathrm{T}})$ All combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$ A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f / \partial x_i \partial x_j =$ **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$ Appears in block elimination on $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Schwarz inequality $|v \cdot w| \le \|v\| \|w\|$ Then $|v^{\mathrm{T}} A w|^2 \le (v^{\mathrm{T}} A v)(w^{\mathrm{T}} A w)$ if $A = C^{\mathrm{T}} C$.

Semidefinite matrix A (Positive) semidefinite means symmetric with $x^{\mathrm{T}} A x \ge 0$ for all vectors x. Then all eigenvalues $\lambda \ge 0$; no negative pivots.

Similar matrices A **and** B $B = M^{-1} A M$ has the same eigenvalues as A.

Simplex method for linear programming The minimum cost vector x^* is found by moving from corner to lower-cost corner along the edges of the feasible set (where the constraints $Ax = b$ and $x \ge 0$ are satisfied). Minimum cost at a corner!

Singular matrix A A square matrix that has no inverse: $\det(A) = 0$.

Singular Value Decomposition (SVD) $A = U \Sigma V^{\mathrm{T}} =$ **(orthogonal** U**) times (diagonal** Σ**) times (orthogonal** V^{T}**)** First r columns of U and V are orthonormal bases of $\boldsymbol{C}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$, with $Av_i = \sigma_i u_i$ and singular value $\sigma_i > 0$. Last columns of U and V are orthonormal bases of the nullspaces of A^{T} and A.

Skew-symmetric matrix K The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $Ax = b$ The right side b is in the column space of A.

Spanning set $v_1, \ldots, v_m$**, for V** Every vector in $\mathbf{V}$ is a combination of $v_1, \ldots, v_m$.

Special solutions to $As = 0$ One free variable is $s_i = 1$, other free variables $= 0$.

Spectral theorem $A = Q \Lambda Q^{\mathrm{T}}$ Real symmetric A has real λ_i and orthonormal q_i, with $Aq_i = \lambda_i q_i$. In mechanics, the q_i give the *principal axes*.

Spectrum of A The set of eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. **Spectral radius** $= |\lambda_{\max}|$.

Standard basis for $\mathbf{R}^n$ Columns of n by n identity matrix (written i, j, k in $\mathbf{R}^3$).

Stiffness matrix K When x gives the movements of the nodes in a discrete structure, Kx gives the internal forces. Often $K = A^{\mathrm{T}} C A$, where C contains spring constants from Hooke's Law and $Ax =$ stretching (strains) from the movements x.

Subspace S of V Any vector space inside **V**, including **V** and $\mathbf{Z} = \{\text{zero vector}\}$.

Sum $\mathbf{V}+\mathbf{W}$ of subspaces Space of all (v in V)+(w in **W**). **Direct sum**: $\dim(\mathbf{V}+\mathbf{W}) = \dim \mathbf{V} + \dim \mathbf{W}$, when **V** and **W** share only the zero vector.

Symmetric factorizations $A = LDL^{\mathrm{T}}$ and $A = Q\Lambda Q^{\mathrm{T}}$ The number of positive pivots in D and positive eigenvalues in Λ is the same.

Symmetric matrix A The transpose is $A^{\mathrm{T}} = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric. All matrices of the form $R^{\mathrm{T}}R$ and LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are symmetric. Symmetric matrices have real eigenvalues in Λ and orthonormal eigenvectors in Q.

Toeplitz matrix T Constant-diagonal matrix, so t_{ij} depends only on $j-i$. Toeplitz matrices represent linear time-invariant filters in signal processing.

Trace of A Sum of diagonal entries = sum of eigenvalues of A. $\mathrm{Tr}AB = \mathrm{Tr}BA$.

Transpose matrix A^{T} Entries $A^{\mathrm{T}}_{ij} = A_{ji}$. A^{T} is n by m, $A^{\mathrm{T}}A$ is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are $B^{\mathrm{T}}A^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$.

Triangle inequality $\|u+v\| \le \|u\| + \|v\|$ For matrix norms, $\|A+B\| \le \|A\| + \|B\|$.

Tridiagonal matrix T $t_{ij} = 0$ if $|i-j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}} = U^{-1}$ Orthonormal columns (complex analog of Q).

Vandermonde matrix V $Vc = b$ gives the polynomial $p(x) = c_0 + \cdots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$ at n points. $V_{ij} = (x_i)^{j-1}$, and $\det V$ = product of $(x_k - x_i)$ for $k > i$.

Vector addition $v + w = (v_1 + w_1, \ldots, v_n + w_n)$ = diagonal of parallelogram.

Vector space V Set of vectors such that all combinations $cv + dw$ remain in **V**. Eight required rules are given in Section 2.1 for $cv + dw$.

Vector v in $\mathbf{R}^n$ Sequence of n real numbers $v = (v_1, \ldots, v_n)$ = point in $\mathbf{R}^n$.

Volume of box The rows (or columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$ or vectors w_{jk} Rescale and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$. Vectors from $w_{00} = (1, 1, -1, -1)$ would be $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$.