The Discrete Fourier Transform and Simple Filtering

Big Picture

We have been thinking about data as vectors, and about changing coordinates to understand structure.

- In SVD, we chose a basis adapted to the data.
- In PCA, we interpreted those coordinates in terms of variance. Today, we choose a **fixed basis** that is good for signals.

The Discrete Fourier Transform (DFT) is just a change of coordinates.

Filtering becomes simple in those coordinates.

Signals as Vectors

A discrete signal of length N is just a vector

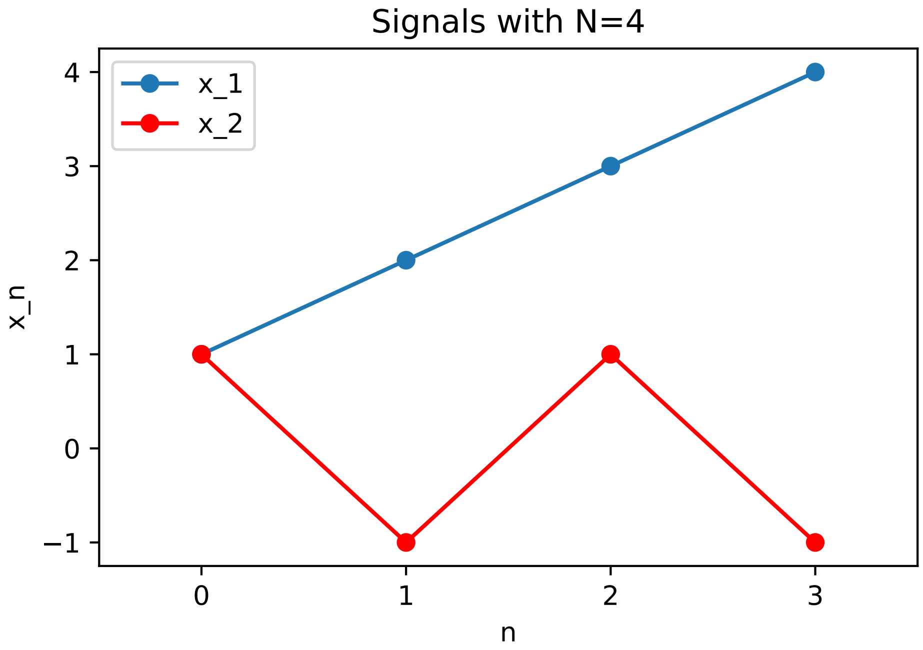
$$x = (x_0, x_1, \dots, x_{N-1})^T \in \mathbb{C}^N.$$

We can think of each point x_n as a sample from a presumably continuous signal.

Examples: for $N = 4$,

$$x_1 = [1, 2, 3, 4], \quad x_2 = [1, -1, 1, -1].$$

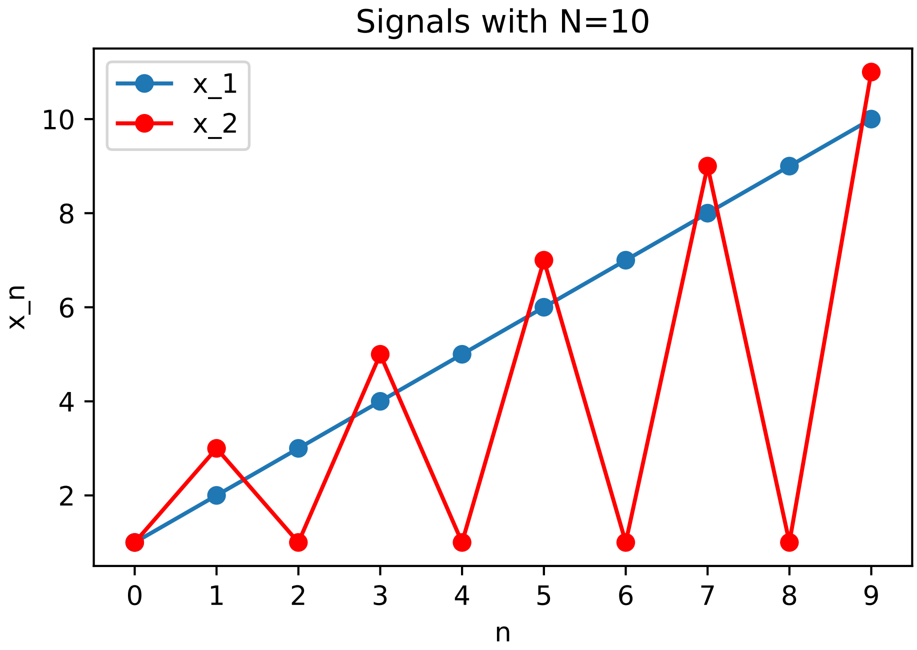
```
import matplotlib.pyplot as plt
import numpy as np

x_1 = np.array([1,2,3,4])
plt.plot(x_1 , marker='o')
x_2 = np.array([1,-1,1,-1])
plt.plot(x_2, marker='o', color='red')
plt.legend(['x_1', 'x_2'])
plt.xlabel("n")
plt.ylabel("x_n")
plt.xlim(-0.5, 3.5)
plt.xticks([0, 1, 2, 3])
plt.title('Signals with N=4')
plt.show()
```



Or for $N = 10$,

$$x = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], \quad x = [1, 3, 1, 5, 1, 7, 1, 9, 1, 11].$$

```
import matplotlib.pyplot as plt
import numpy as np

x_1 = np.array([1,2,3,4,5,6,7,8,9,10])
plt.plot(x_1 , marker='o')
x_2 = np.array([1,3,1,5,1,7,1,9,1,11])
plt.plot(x_2, marker='o', color='red')
plt.legend(['x_1', 'x_2'])
plt.xlabel("n")
plt.ylabel("x_n")
plt.xlim(-0.5, 9.5)
plt.xticks([0, 1, 2, 3, 4, 5, 6, 7, 8, 9])
plt.title('Signals with N=10')
plt.show()
```



Discrete Fourier Transform

Fourier series

The Fourier series of a continuous function $f(t)$ is an expansion into sines and cosines:

$$f(t) = a_0 + a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \cdots$$

. . .

Euler's formula tells us that the complex exponential e^{it} is a combination of $\cos t$ and $\sin t$:

$$e^{it} = \cos t + i \sin t$$

. . .

So we can rewrite the Fourier series using complex exponentials:

$$f(t) = c_0 + c_1 e^{it} + c_2 e^{2it} + \cdots = \sum_k c_k \cdot e^{ikt}$$

. . .

The coefficients $c_0, c_1, c_2, \ldots$ are related to the a's and b's.

For example, $c_1 = a_1 - ib_1$.

Discrete signals

But here we are interested in discrete signals. To connect with the notation from earlier in the lecture, instead of $f(t)$, we have $f[n] = x_n$ for $n = 0, 1, \ldots, N-1$.

. . .

In this case, we have basis vectors

$$f_k[n] = e^{i2\pi kn/N}, \quad \text{for } n = 0, 1, \ldots, N-1.$$

. . .

These are continuous complex exponentials (cosines and sines), but sampled at N points.

We will have one basis vector, or Fourier mode, for each frequency index $k = 0, 1, \ldots, N-1$.

The frequency index k is the number of times the underlying continuous oscillation fits into N samples.

These vectors form an orthogonal basis of $\mathbb{C}^N$.

So every signal can be written as a combination of oscillations.

The Fourier basis vector or mode $\mathbf{f}_k$ is given by

$$\mathbf{f}_k = \begin{bmatrix} e^{i2\pi k\cdot 0/N} \\ e^{i2\pi k\cdot 1/N} \\ \vdots \\ e^{i2\pi k\cdot (N-1)/N} \end{bmatrix}.$$

. . .

Each mode $\mathbf{f}_k$ traces one of the possible discrete oscillations.

- $k = 0 \rightarrow$ constant signal
- small $k \rightarrow$ slow oscillation

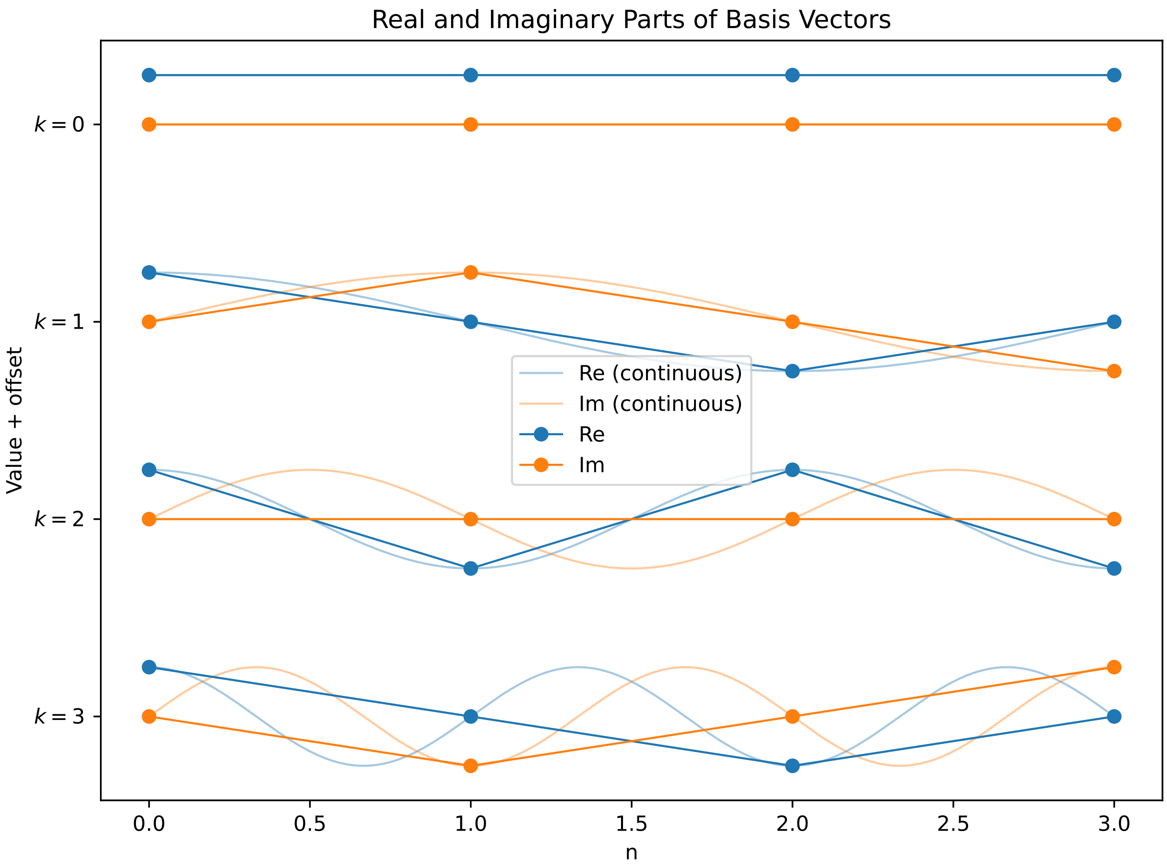
- large k → rapid oscillation
- $k = N/2$ → fastest oscillation, $N/2$ cycles in N samples
- k larger than $N/2$ → oscillation actually slower (will describe with pictures)

Visualizing the Fourier basis vectors

. . .

Let's look at these basis vectors for $N = 4$:

```python
import numpy as np
import matplotlib.pyplot as plt

N = 4
N_fine = 1000
n_fine = np.linspace(0, N - 1, N_fine)
f_k = np.array([np.exp(2j * np.pi * k * np.arange(N) / N) for k in range(N)])

plt.figure(figsize=(8, 6))
offsets = np.arange(N, 0, -1)*4  # 10, 9, ..., 1

for k in range(N):
    # Continuous curves (as if sampled at N_fine points)
    re_continuous = np.cos(2 * np.pi * k * n_fine / N) + offsets[k]
    im_continuous = np.sin(2 * np.pi * k * n_fine / N) + offsets[k]
    plt.plot(n_fine, re_continuous, color='C0', alpha=0.4, linewidth=1, zorder=0, label='Re
    plt.plot(n_fine, im_continuous, color='C1', alpha=0.4, linewidth=1, zorder=0, label='Im
    # Discrete samples
    y_real = f_k[k].real + offsets[k]
    y_imag = f_k[k].imag + offsets[k]
    plt.plot(np.arange(N), y_real, marker='o', color='C0', linewidth=1, zorder=1, label='Re'
    plt.plot(np.arange(N), y_imag, marker='o', color='C1', linewidth=1, zorder=2, label='Im'
# Set y-ticks to correspond to the offsets so they show k values
plt.yticks(offsets, [f'$k={k}$' for k in range(N)])

plt.xlabel('n')
plt.ylabel('Value + offset')
plt.title('Real and Imaginary Parts of Basis Vectors')
plt.legend()
plt.tight_layout()
plt.show()
```



The four vectors given by the blue (real) and orange (imaginary) points form a basis of $\mathbb{C}^4$:

```
# Calculate and display the explicit values of the DFT basis vectors for N=4 using sympy for
import sympy as sp

N = 4
k_vals = list(range(N))
n_vals = list(range(N))
omega = sp.exp(2 * sp.pi * sp.I / N)  # fundamental N-th root of unity

# Build the matrix of basis vectors: each row k is [omega^(k n) for n=0..N-1]
F = sp.Matrix(N, N, lambda k, n: omega**(k * n))
F
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

(Each column k of this matrix is $\mathbf{f}_k$, the k-th basis vector.)

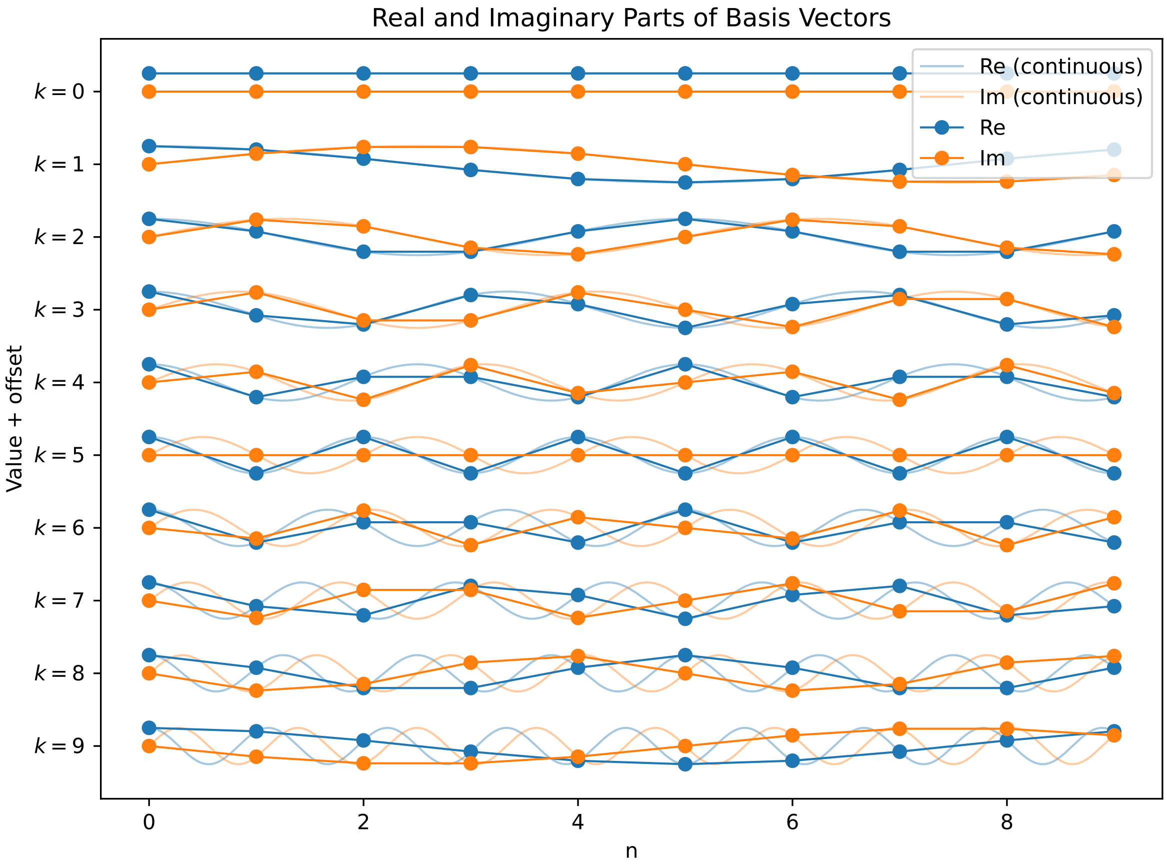
. . .

For N=10

. . .

Let's look at these basis vectors for $N = 4$:

```
import numpy as np
import matplotlib.pyplot as plt

N = 10
N_fine = 1000
n_fine = np.linspace(0, N - 1, N_fine)
f_k = np.array([np.exp(2j * np.pi * k * np.arange(N) / N) for k in range(N)])

plt.figure(figsize=(8, 6))
offsets = np.arange(N, 0, -1)*4  # 10, 9, ..., 1

for k in range(N):
    # Continuous curves (as if sampled at N_fine points)
    re_continuous = np.cos(2 * np.pi * k * n_fine / N) + offsets[k]
    im_continuous = np.sin(2 * np.pi * k * n_fine / N) + offsets[k]
    plt.plot(n_fine, re_continuous, color='C0', alpha=0.4, linewidth=1, zorder=0, label='Re
    plt.plot(n_fine, im_continuous, color='C1', alpha=0.4, linewidth=1, zorder=0, label='Im
    # Discrete samples
    y_real = f_k[k].real + offsets[k]
    y_imag = f_k[k].imag + offsets[k]
    plt.plot(np.arange(N), y_real, marker='o', color='C0', linewidth=1, zorder=1, label='Re'
    plt.plot(np.arange(N), y_imag, marker='o', color='C1', linewidth=1, zorder=2, label='Im'
# Set y-ticks to correspond to the offsets so they show k values
plt.yticks(offsets, [f'$k={k}$' for k in range(N)])

plt.xlabel('n')
plt.ylabel('Value + offset')
plt.title('Real and Imaginary Parts of Basis Vectors')
plt.legend()
plt.tight_layout()
plt.show()
```



Expressing a signal

OK, now we are back in familiar territory!

We have a basis of vectors, and we can express any signal as a combination of these vectors just by finding the projections onto each of them.

. . .

So if we have a signal $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]$, we can find the coefficients (call them X_k) by taking the dot product of $\mathbf{x}$ with $\mathbf{f}_k$:

$$X_k = \mathbf{x} \cdot \mathbf{f}_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N}.$$

. . .

This is the Discrete Fourier Transform (DFT)!

DFT measures frequency content

The DFT gives the coordinates of $\mathbf{x}$ in an oscillatory basis

Conceptually:

- X_k measures "how much frequency k is present."
- The DFT is just a coordinate change.

Inverse DFT

The inverse DFT reconstructs the signal, given the coefficients X_k:

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}.$$

Relationship to the Fourier series

For continuous functions, we can find the coefficients by integrating:

$$c_k = \int_0^{2\pi} f(t) e^{-ikt} dt$$

This is equivalent to taking the *projection* of $f(t)$ onto the basis function e^{ikt}.

. . .

Here instead we have

$$X_k = \mathbf{x} \cdot \mathbf{f}_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N}.$$

. . .

We do summation instead of integration because both our signal and our basis functions are discrete.

. . .

Example: Computing a Small DFT by Hand ($N = 4$)

Computing a Small DFT by Hand ($N = 4$)

Let's find the DFT of the signal $\mathbf{x} = [1, -1, 1, -1]$.

. . .

We define a constant just to make the algebra less messy.

When $N = 4$, let

$$W = e^{-i2\pi/4} = -i.$$

. . .

So the formula for the DFT becomes

$$X_k = \sum_{n=0}^{3} x_n W^{kn}.$$

. . .

Note that the powers of W cycle:

$$W^0 = 1, \quad W^1 = -i, \quad W^2 = -1, \quad W^3 = i \quad \text{and then we wrap to} \quad W^4 = 1, \quad W^5 = -i, \quad \text{and so on.}$$

$$\mathbf{x} = [1, -1, 1, -1], \qquad X_k = \sum_{n=0}^{3} x_n W^{kn}.$$

$$W^0 = 1, \quad W^1 = -i, \quad W^2 = -1, \quad W^3 = i.$$

. . .

So for X_0, we have

$$X_0 = x_0 W^{0\cdot 0} + x_1 W^{0\cdot 1} + x_2 W^{0\cdot 2} + x_3 W^{0\cdot 3}$$

$$= 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 1 + (-1) \cdot 1 = 0.$$

. . .

X_1:

$$\begin{aligned} X_1 &= x_0 W^{1\cdot 0} + x_1 W^{1\cdot 1} + x_2 W^{1\cdot 2} + x_3 W^{1\cdot 3} \\ &= 1 \cdot 1 + (-1) \cdot (-i) + 1 \cdot (-1) + (-1) \cdot i \\ &= 1 + i - 1 - i = 0. \end{aligned}$$

Also zero!

X_2:

$$\begin{aligned} X_2 &= x_0 W^{2\cdot 0} + x_1 W^{2\cdot 1} + x_2 W^{2\cdot 2} + x_3 W^{2\cdot 3} \\ &= 1 \cdot 1 + (-1) \cdot (-1) + 1 \cdot 1 + (-1) \cdot (-1) \\ &= 4. \end{aligned}$$

This one is non-zero!

. . .

Finally, for X_3:

$$\begin{aligned}X_3 &= x_0W^{3\cdot 0} + x_1W^{3\cdot 1} + x_2W^{3\cdot 2} + x_3W^{3\cdot 3}\\ &= 1\cdot 1 + (-1)\cdot i + 1\cdot(-1) + (-1)\cdot(-i)\\ &= 0.\end{aligned}$$

Zero again.

Putting these together, we get

$$X = [0, 0, 4, 0].$$

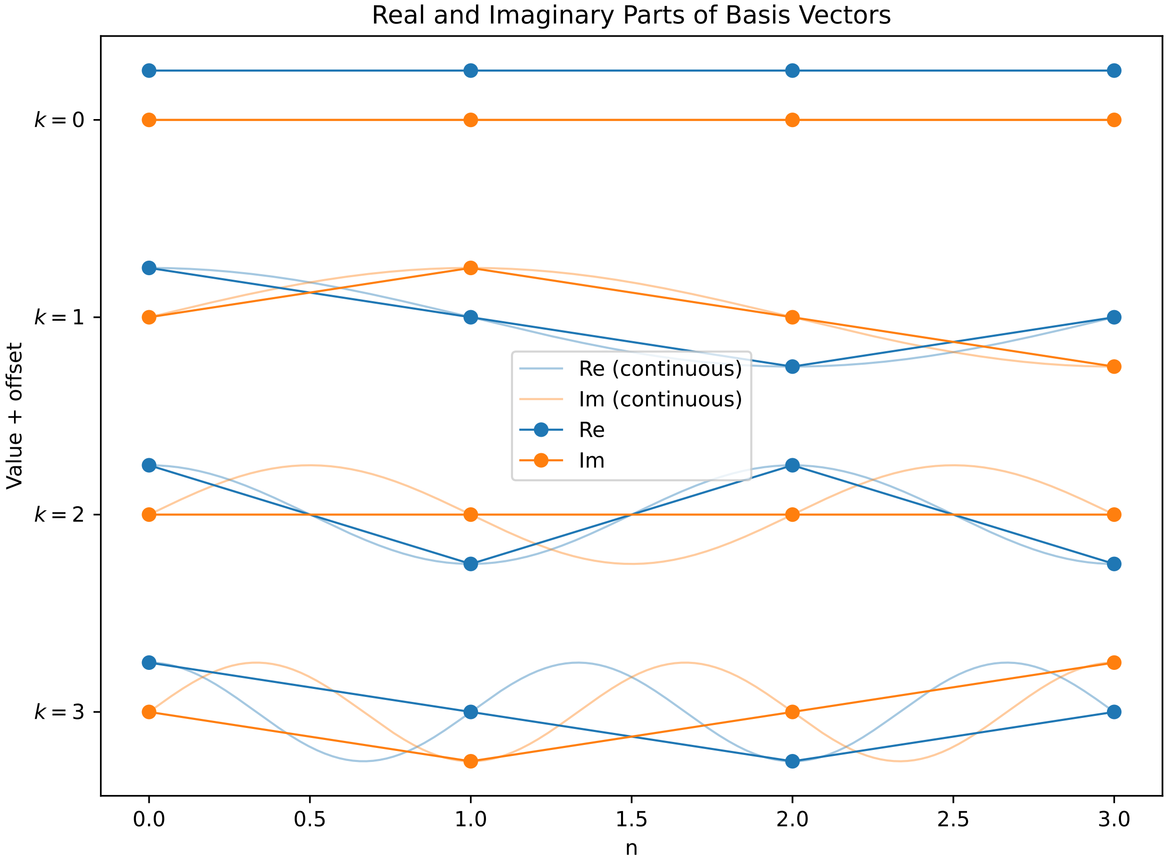
$$x = [1, -1, 1, -1], \quad X = [0, 0, 4, 0].$$

Interpretation:

All the energy is a in the mode $k = 2$.

. . .

```
import numpy as np
import matplotlib.pyplot as plt

N = 4
N_fine = 1000
n_fine = np.linspace(0, N - 1, N_fine)
f_k = np.array([np.exp(2j * np.pi * k * np.arange(N) / N) for k in range(N)])

plt.figure(figsize=(8, 6))
offsets = np.arange(N, 0, -1)*4  # 10, 9, ..., 1

for k in range(N):
    # Continuous curves (as if sampled at N_fine points)
    re_continuous = np.cos(2 * np.pi * k * n_fine / N) + offsets[k]
    im_continuous = np.sin(2 * np.pi * k * n_fine / N) + offsets[k]
    plt.plot(n_fine, re_continuous, color='C0', alpha=0.4, linewidth=1, zorder=0, label='Re
    plt.plot(n_fine, im_continuous, color='C1', alpha=0.4, linewidth=1, zorder=0, label='Im
    # Discrete samples
    y_real = f_k[k].real + offsets[k]
    y_imag = f_k[k].imag + offsets[k]
    plt.plot(np.arange(N), y_real, marker='o', color='C0', linewidth=1, zorder=1, label='Re'
    plt.plot(np.arange(N), y_imag, marker='o', color='C1', linewidth=1, zorder=2, label='Im'
# Set y-ticks to correspond to the offsets so they show k values
plt.yticks(offsets, [f'$k={k}$' for k in range(N)])

plt.xlabel('n')
plt.ylabel('Value + offset')
plt.title('Real and Imaginary Parts of Basis Vectors')
plt.legend()
plt.tight_layout()
plt.show()
```



. . .

This signal alternates every sample point — it is the highest possible frequency for $N = 4$.

Filters on Discrete Signals

Filters on Signals

We've seen filters before. They are simply linear transformations that modify the signal, acting as they slide along the signal.

. . .

For instance, we looked at a filter that took a weighted average of three adjacent points in the signal. The output point was given by

$$y_n = \frac{1}{2}(\frac{1}{2}x_{n-2} + x_{n-1} + \frac{1}{2}x_n).$$

. . .

(Here we only use points at n and to its left, which is a common convention in signal processing.)

. . .

Filter notation

$$y_n = \frac{1}{2}(\frac{1}{2}x_{n-2} + x_{n-1} + \frac{1}{2}x_n).$$

We write this filter as $h = \frac{1}{2}([\frac{1}{2}, 1, \frac{1}{2}])$.

h is a vector of coefficients, describing the effect of the filter at each point in the sliding window.

. . .

The **transformation** of applying the filter h to a signal $\mathbf{x}$ is written as $\mathscr{F}_h$:

$$\mathbf{y} = \mathscr{F}_h(\mathbf{x}).$$

(This is not a standardized notation, just one I picked for this lecture.)

Example: Simple 2-point filter

Here, we are going to look at an even simpler filter, which will just take the average of *two* adjacent points: x_{n-1} and x_n.

. . .

$y_n = \frac{1}{2}(x_{n-1} + x_n)$

. . .

Filtering as matrix multiplication

If we know the length of the signal, we can write the filter as a matrix multiplication.

. . .

One small gotcha: we need to decide what to do with the points at the edges of the signal. Here we will assume that the signal wraps around, so $x_{-1} = x_5$.

. . .

For our filter $h = \frac{1}{2}([\frac{1}{2}, 1])$, we have

$$\mathbf{y} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}\mathbf{x}.$$

Side note: making the filter longer

$y_n = \frac{1}{2}(x_{n-1} + x_n)$

. . .

I am going to write down a longer filter, which has the exact same effect as this filter. (This is just going to be useful in the next slide or two!)

. . .

Here it is:

$$y_n = 0x_{n-4} + 0x_{n-3} + 0x_{n-2} + \tfrac{1}{2}x_{n-1} + \tfrac{1}{2}x_n$$

. . .

In other words, we have $h = [\frac{1}{2}, \frac{1}{2}, 0, 0, 0]$.

. . .

The only reason I did this is that we are about to work with sums, and it's a bit confusing to have a sum of only length 2.

A compact way to describe the filter's action

$$y_n = 0x_{n-4} + 0x_{n-3} + 0x_{n-2} + \tfrac{1}{2}x_{n-1} + \tfrac{1}{2}x_n; \quad h = [0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}].$$

. . .

Rewriting:

$$y_n = h_5 x_{n-5} + h_4 x_{n-4} + h_3 x_{n-3} + h_2 x_{n-2} + h_1 x_{n-1} + h_0 x_n = \sum_{m=0}^{L-1} h_m x_{n-m}.$$

. . .

This now is the definition of how a filter acts on a signal:

$$y_n = \sum_{m=0}^{L-1} h_m x_{n-m}.$$

Filtering as convolution

$$y_n = \sum_{m=0}^{L-1} h_m x_{n-m}.$$

. . .

This is actually a mathematical operation called **convolution**.

. . .

When we take the **convolution** of $\mathbf{x}$ with $\mathbf{h}$, it is defined as:

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \implies y_n = \sum_{m=0}^{L-1} h_m x_{n-m}.$$

. . .

This means that the filter transformation is actually a convolution:

$$\mathcal{F}_h(\mathbf{x}) = \mathbf{x} \circledast \mathbf{h}.$$

. . .

So we can specify a filter by a vector $\mathbf{h} = [h_0, h_1, \dots, h_{L-1}]$, and apply it to a signal $\mathbf{x}$ by convolving them.

Summary so far

Signals: - We write a signal $\mathbf{x}$ as a vector of samples: $\mathbf{x} = [x_0, x_1, \dots, x_{N-1}]$.

- We can express this signal as a weighted sum of Fourier modes: $\mathbf{x} = \sum_{k=0}^{N-1} X_k \mathbf{f}_k$, where each Fourier mode is given by $\mathbf{f}_k[n] = e^{i2\pi kn/N}$.

Filters: - We can specify a filter by a vector $\mathbf{h} = [h_0, h_1, \dots, h_{L-1}]$

- The filter's action on a signal $\mathbf{x}$ is given by the formula

$$y_n = \sum_{m=0}^{L-1} h_m x_{n-m}.$$

- More compactly, we can write the filter's action as a convolution:

$$\mathbf{y} = \mathcal{F}_h(\mathbf{x}) = \mathbf{x} \circledast \mathbf{h}.$$

Understanding the effects of filters

Understanding the effects of filters

Can we characterize the effects of a given filter?

. . .

We might start by looking at what the filter does to an example signal.

. . .

A warning: This is not going to be the approach we will end up with!

. . .

Our filter: $h = [\frac{1}{2}, \frac{1}{2}]$

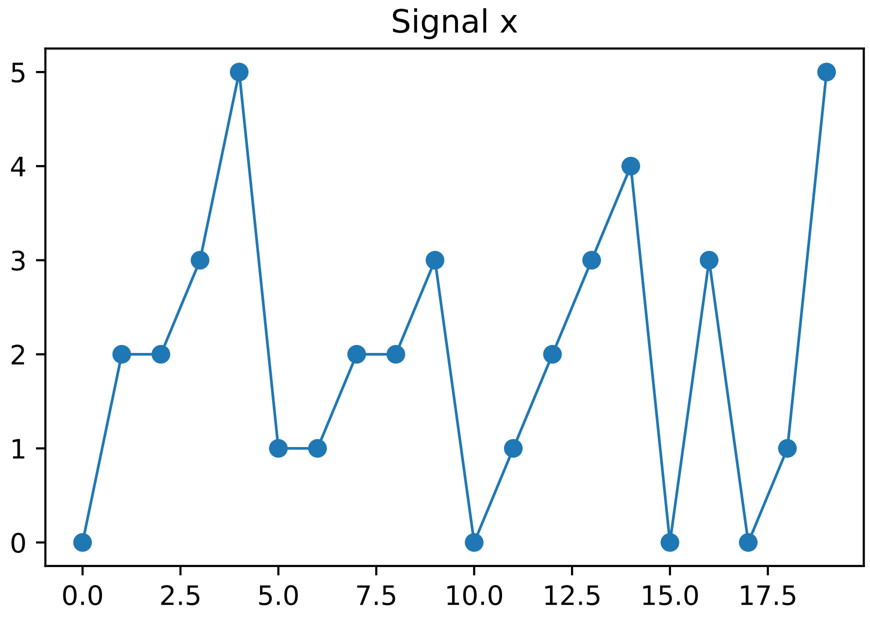
An example signal:

Let's use a sample signal of length 20:

```
import numpy as np

pattern = np.array([1, 2, 1, 2, 4])
x = np.tile(pattern, 4)  # Length 20
# add some integer noise
x += np.random.randint(-1, 2, 20)
print(x)
```

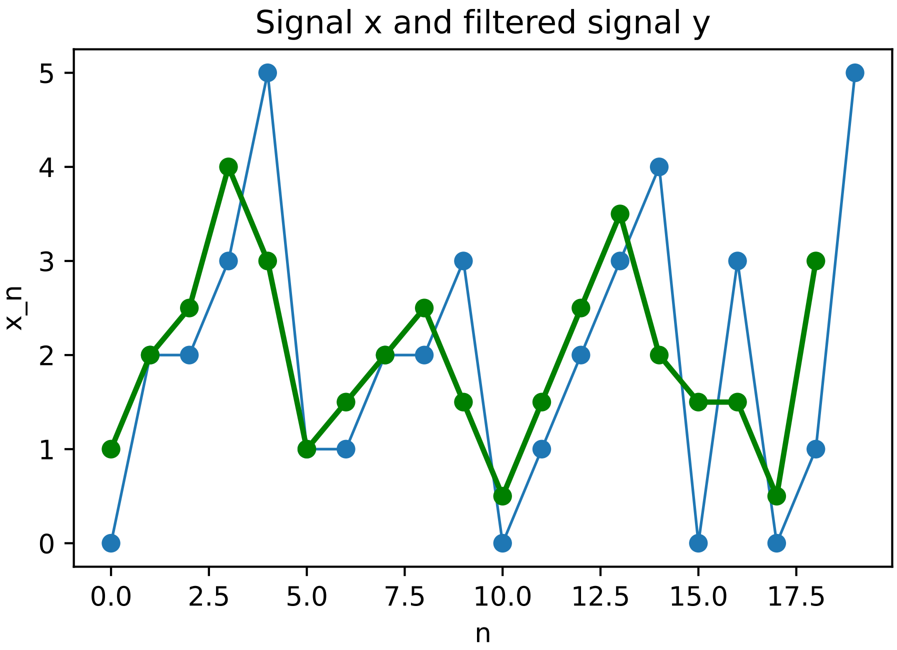
```
[0 2 2 3 5 1 1 2 2 3 0 1 2 3 4 0 3 0 1 5]
```

```
import numpy as np
import matplotlib.pyplot as plt

x = x.astype(int)
N=len(x)
# Plot all points in default color
plt.plot(np.arange(N), x, marker='o', color='C0', linewidth=1, zorder=1)

# Highlight the m
plt.title('Signal x')
plt.show()
```



Then we apply the filter $h = [\frac{1}{2}, \frac{1}{2}]$.

```
import numpy as np
import matplotlib.pyplot as plt

# x is the original signal

# y is the signal filtered by h = 1/2, 1/2

# Compute the filtered signal y by averaging adjacent pairs (length N-1)
h = np.array([0.5, 0.5])
y = np.convolve(x, h, mode='valid')


# Plot all points in default color
plt.plot(x, marker='o', color='C0', linewidth=1, zorder=1)
plt.plot(y, marker='o', color='green', linewidth=2, zorder=2)
# Highlight the middle 4 points (indices 4-7) in red


plt.xlabel("n")
plt.ylabel("x_n")
plt.title('Signal x and filtered signal y')
plt.show()
```



$$y_n = \sum_{m=0}^{1} h_m x_{n-m} = \frac{1}{2}x_{n-1} + \frac{1}{2}x_n$$

Hmm. It's actually kind of hard to see what the filter is doing to this signal. . . .

But can we be more systematic about this?

Finding a special basis for signals

We understood the actions of matrix multiplication by finding vectors which behaved simply under the multiplication.

. . .

$$A\mathbf{v} = \lambda\mathbf{v}$$

. . .

These were, of course, the eigenvectors of A.

. . .

Then if made a basis out of those eigenvectors, we could understand the action of A on *any* vector $\mathbf{x}$ by expressing $\mathbf{x}$ in this basis.

. . .

Can we find something equivalent for signals and filters? That is, will there be a basis of *signals* where the action of a filter is made simple?

. . .

The answer is yes! And the basis we want will be none other than the DFT basis.

What Does a Filter Do to a Fourier Mode?

Let us take as our $\mathbf{x}$ one of the Fourier modes, basis vectors of the DFT, $f_k[n] = e^{i2\pi kn/N}$.

. . .

For simplicity, define $\zeta_k = 2\pi k/N$. Then

$$f_k[n] = e^{i\zeta_k n}.$$

. . .

What happens when we apply the filter $h = [\frac{1}{2}, \frac{1}{2}]$ to this signal?

We use convolution to see what happens when we apply the filter to the k-th Fourier mode:

$$\mathbf{y}_k = \mathcal{F}_h(f_k) = f_k \circledast h.$$

$$y_k[n] = \sum_{m=0}^{L-1} h_m f_k[n-m] = \sum_{m=0}^{L-1} h_m e^{i\zeta_k(n-m)} = \left(\sum_{m=0}^{L-1} h_m e^{-i\zeta_k m} \right) e^{i\zeta_k n}.$$

Simplifying the expression

$$y_k[n] = \sum_{m=0}^{L-1} h_m f_k[n-m] = \sum_{m=0}^{L-1} h_m e^{i\zeta_k(n-m)} = \left(\sum_{m=0}^{L-1} h_m e^{-i\zeta_k m} \right) e^{i\zeta_k n}.$$

. . .

But wait. This simplifies!

The part inside the parentheses is just a constant, and it doesn't depend at all on n. Let's just give it a name, $H(\zeta_k)$.

. . .

$$H(\zeta_k) \equiv \sum_{m=0}^{L-1} h_m e^{-i\zeta_k m}.$$

So

$$y_k[n] = H(\zeta_k) f_k[n].$$

This holds at every point n, so we can write it in vector form as

$$\mathbf{y}_k = H(\zeta_k)\mathbf{f}_k.$$

. . .

This means that if our input is one of the Fourier modes, the output is simply the input scaled by $H(\zeta_k)$.

. . .

The Fourier modes are the eigenvectors of the filter!

Fourier modes as eigenvectors

$$\mathbf{y}_k = H(\zeta_k)\mathbf{f}_k; \quad H(\zeta_k) = \sum_{m=0}^{L-1} h_m e^{-i\zeta_k m} \quad \text{for} \quad k = 0, 1, \dots, N-1.$$

. . .

This derivation didn't depend on the form of the filter h.

. . .

That means that the Fourier modes are the eigenvectors of *every* filter that we can write as a convolution!

Behavior on any signal

We can now use the DFT to understand the behavior of a filter on any signal.

. . .

Remember, if we have a signal $\mathbf{x}$, we can write it as a weighted sum of the DFT basis vectors:

$$\mathbf{x} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \mathbf{f}_k.$$

. . .

Since our filter is linear, we can apply it to each of the terms in this sum separately:

$$\mathcal{F}_h(\mathbf{x}) = \mathcal{F}_h\left(\frac{1}{N}\sum_{k=0}^{N-1} X_k \mathbf{f}_k\right) = \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{F}_h(X_k \mathbf{f}_k)$$

. . .

$$\mathscr{F}_h(\mathbf{x}) = \frac{1}{N}\sum_{k=0}^{N-1} \mathscr{F}_h(X_k \mathbf{f}_k)$$

Since X_k is a constant, we can factor it out of the transformation:

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k \mathscr{F}_h(\mathbf{f}_k)$$

. . .

Now we can use the fact that the Fourier modes are the eigenvectors of the filter, so $\mathscr{F}_h(\mathbf{f}_k) = H(\zeta_k)\mathbf{f}_k$, to write:

$$\mathscr{F}_h(\mathbf{x}) = \frac{1}{N}\sum_{k=0}^{N-1} X_k H(\zeta_k)\mathbf{f}_k = \frac{1}{N}\sum_{k=0}^{N-1} H(\zeta_k) X_k \mathbf{f}_k$$

. . .

So we can describe the action of the filter: it takes each Fourier mode k and scales it by $H(\zeta_k)$.

$$\mathscr{F}_h(\mathbf{x}) = \frac{1}{N}\sum_{k=0}^{N-1} X_k H(\zeta_k)\mathbf{f}_k = \frac{1}{N}\sum_{k=0}^{N-1} H(\zeta_k) X_k \mathbf{f}_k$$

Steps to understanding what a filter does to a signal of length N:

1. Find the DFT of the signal. This will decompose the signal into a weighted sum of N Fourier modes $\mathbf{f}_k$, with weights X_k.
2. For each k, compute $H(\zeta_k)$ for the filter h.
3. Multiply each Fourier mode $\mathbf{f}_k$ by X_k and $H(\zeta_k)$ and sum the results to get the output.

. . .

Interpretation:

- The filter scales each Fourier mode by a complex number.

. . .

What does it mean to scale a mode by a complex number? We will look at an example to understand.

Analyzing two simple filters

Lowpass: Moving Average

$$h = \left[\tfrac{1}{2}, \tfrac{1}{2}\right].$$

Then we calculate the frequency response using the formula we derived:

$$H(\zeta) = \sum_{m=0}^{L-1} h_m e^{-i\zeta m} = \tfrac{1}{2}(e^0 + e^{-i\zeta}) = \tfrac{1}{2}(1 + e^{-i\zeta}).$$

Interlude: Complex Exponentials

Our frequency response is:

$$H(\zeta) = \tfrac{1}{2}(1 + e^{-i\zeta}).$$

. . .

Let's pause here. We will get something like this for every filter we analyze – something that's a sum of complex exponentials.

. . .

Complex exponentials are vectors in the complex plane, and they have **magnitude** and **angles** (or **phases**).

. . .

For our purposes here, we want to be able to find the magnitudes. And in general this is a bit complicated!

. . .

Sometimes, there are simple trigenometric identities that can help. We will use one on the next slide.

. . .

But often, we only end up wanting to know the magnitude for certain values of ζ like 0 and π – and this turns out to simplify things even more!

$$H(\zeta) = \tfrac{1}{2}(1 + e^{-i\zeta}).$$

We factor out $e^{-i\zeta/2}$:

$$H(\zeta) = \tfrac{1}{2}e^{-i\zeta/2}\left(e^{i\zeta/2} + e^{-i\zeta/2}\right).$$

$$H(\zeta) = \tfrac{1}{2}e^{-i\zeta/2}\left(e^{i\zeta/2} + e^{-i\zeta/2}\right).$$

. . .

Recall Euler's identity:

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta).$$

. . .

Apply this with $\theta = \zeta/2$:

$$e^{i\zeta/2} + e^{-i\zeta/2} = 2\cos(\zeta/2).$$

. . .

So

$$H(\zeta) = \tfrac{1}{2}e^{-i\zeta/2}\left(2\cos(\zeta/2)\right) = e^{-i\zeta/2}\cos(\zeta/2).$$

Finding the Magnitude

$$H(\zeta) = e^{-i\zeta/2}\cos(\zeta/2).$$

Now we clearly see two pieces:

- A **phase**, which is a complex number of magnitude 1: $e^{-i\zeta/2}$

(Every single complex exponential of the form $e^{i\theta}$ has magnitude 1. *Sums* of complex exponentials don't necessarily have magnitude 1, though!)

- A scaling factor or **gain**, which is a real number: $\cos(\zeta/2)$

. . .

Notice that the gain is just the magnitude of the complex number $H(\zeta)$:

$$|H(\zeta)| = \cos(\zeta/2).$$

Writing the frequency response in terms of k

. . .

Remembering our definition of $\zeta_k = 2\pi k/N$, we have

Phase shift: $e^{-i\zeta_k/2} = e^{-i\pi k/N}$.

Gain: $\cos(\zeta_k/2) = \cos(\pi k/N)$.

. . .

The action of the filter on a Fourier mode k is to *shift* the phase by $e^{-i\pi k/N}$ and to *scale* the amplitude by $\cos(\pi k/N)$.

Checking the Endpoints

$$|H(\zeta_k)| = |\cos(\pi k/N)|.$$

Let's understand what the filter does at the lowest possible frequency and at the highest possible frequency.

. . .

At the *lowest* frequency, $k = 0$:

$$|H(0)| = |\cos(0)| = 1.$$

. . .

At the highest frequency (e.g. $k = N/2$ when N is even), $\zeta = \pi$:

$$|H(\pi)| = |\cos(\pi/2)| = 0.$$

So:

- Slow oscillations (small k) pass through.
- Rapid oscillations (near $k = N/2$) are suppressed.

That's exactly what a lowpass filter does.

Highpass: Difference Filter

Let's go through the same process for a highpass filter.

$$h = \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

. . .

Then

$$H(\zeta) = -\tfrac{1}{2} + \tfrac{1}{2}e^{-i\zeta}.$$

. . .

Rewrite:

$$H(\zeta) = e^{-i\zeta/2}(-i\sin(\zeta/2)).$$

. . .

Find the magnitude of the frequency response:

$$|H(\zeta)| = |\sin(\zeta/2)|.$$

$$|H(\zeta)| = |\sin(\zeta/2)|.$$

Check high and low frequencies:

$$|H(0)| = 0, \qquad |H(\pi)| = 1.$$

Low frequencies are suppressed. Rapid changes are emphasized.

This is a highpass filter.

Final Conceptual Summary

The DFT is just a coordinate change:

- Time domain → frequency domain.
- Convolution → multiplication.
- Filtering → scaling oscillations.

For the homework, you only need:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N},$$

$$H(\zeta) = \sum_{m=0}^{L-1} h_m e^{-i\zeta m},$$

and the idea that filters act differently on different frequencies.

Everything else is interpretation.